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Economic growth models: symbolic and numerical computations

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Abstract

Growth economic models play a crucial role in understanding countries development, inter-country macroeconomic relationship and, ultimately, to anticipate the effects on endogenous variables due to political shocks on model parameters or exogenous variables. Yet, build a mathematical model can be a difficult and time consuming task. Symbolic computations can be of great help in the development process. Then, the ability to simulate, under initial assumptions, is, assuredly, a priceless tool for policy makers to take decisions and to adapt them along the time. Robust and efficient solvers are required to allow for reliable and fast answers. This paper deals with an integrated computational approach to economic growth models, based on the exceptional ability of MATLAB's numerical computing and exploring its symbolic computing capabilities. Illustration is made with the Ramsey-Cass-Koopmans model, one of the macroeconomic workhorse models.

Keywords: *Economic growth, symbolic computations, numerical computations.*

1. Introduction

The Ramsey-Cass-Koopmans (RCK) is a cornerstone in neoclassical growth theory. Cass (1965) and Koopmans (1965) combined the maximization for an infinite horizon, suggested by Ramsey [5] with the Solow-Swan's capital accumulation. In this latter, the savings rate is considered exogenous while for the RCK model it is endogenously computed through a consumer optimization problem. It aims at studying whether the accumulation of capital accounts for the long term growth. This is accomplished by modelling the intertemporal allocation of income, i.e., the relation between consumptions and savings, focusing in the dynamics.

Economic models, to fit as much as possible to reality, are complex and in general without closed solution. Numerical methods to approximate the sough solution are the ultimate answer. MATLAB[1] is used both as a tool to assist the development of the model, using the symbolic toolbox based on MuPAD[2] and to solve the model profiting from its state-of-the-art numerical methods. The MuPAD Notebook app can be used as an interactive environment for performing symbolic computations, using the MuPAD language, and then to generate MATLAB code. Another possibility is to use MATLAB functions in the Symbolic Math Toolbox in an integrated way along with the numerical computations. Noteworthy, there are differences between MATLAB and MuPAD syntaxes.

2. The model: brief presentation

The model represents an economy of one sector where households and firms, with optimizer behaviors, interact in competitive markets. Each family chooses “consumption and saving to maximize their dynastic utility, subject to an intertemporal budget constrain” (Ramsey, 1928). The development of the RCK model is performed within the following market economy environment: (i) households provide labor services in exchange for wages, consume and accumulate assets, and (ii) firms have technical know-how to turn inputs into output, rent capital from consumers and hire labor services.

The pillars are infinite horizon model in continuous time, neoclassical assumptions[3], homogeneous families and lack of market failures. The Kaldor stylized facts (revisited in [3]), a set of empirical long term regularities about economic growth, must be incorporated in the modelling process.

[1] MATLAB is a high-level language and interactive environment for numerical computation, visualization, and programming. MATLAB is a trademark of The MathWorks, Inc.

[2] MuPAD stands for Multi-Processing Algebra Data and it is the computer algebra system (CAS) included in MATLAB's Symbolic Math Toolbox.

[3] A neoclassic production function is one with scale constant income, positive but decreasing marginal productivity of inputs, and meets the Inada conditions (see footnote 4).

2.1 Households

Consider a total population at instant t of $L(t) = e^{nt}L(0)$, $L(0) = 1$ where $n = \frac{\dot{L}(t)}{L(t)} \equiv \frac{\frac{dL(t)}{dt}}{L(t)}$ is the labor growth rate. Each household has a lifetime utility given by

$$\int_0^{\infty} u(c(t))e^{-\rho t}L(t)dt = \int_0^{\infty} u(c(t))e^{-(\rho-n)t}dt \quad (1)$$

where ρ is the subjective discount rate, $\rho - n$ the effective discount rate and $c(t) = \frac{C(t)}{L(t)}$ is the per capita consumption at t, being $C(t)$ the consumption. Certain assumptions must be considered: (i) u must be concave and (ii) $\rho - n > 0$.
Denoting by $\mathcal{A}(t)$ the asset holdings of the representative household at time t, the following law of motion (flow budget constraint) can be set

$$\dot{\mathcal{A}}(t) = r(t)\mathcal{A}(t) + (w(t) - c(t))L(t) \quad (2)$$

where $r(t)$ is the risk-free market flow rate of return on assets, $w(t)L(t)$ is the flow of labor income earnings of the household and $c(t)L(t)$ is the flow of consumption; furthermore, denoting by $a(t) = \frac{\mathcal{A}(t)}{L(t)}$, per capita assets, we obtain the per capita asset accumulation

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - \mathrm{c}(t) \quad (3)$$

that is, assets per person rises with $w(t) + t(t)a(t)$(t), the per capita income, and falls with $c(t) + na(t)$, per capita consumption and expansion of population. The budget constraint appears by considering market clearing, $a(t) = \frac{K(t)}{L(t)} = k(t)$, that is, the rate of change of assets and of capital per capita must equal (in equilibrium, there is no borrowing/lending between households, and all assets held are claims on capital).
The flow budget constraint (3) allows for a Ponzi game, that is, the agent can increase consumption by taking debt and therefore $\lim_{t\to\infty} a(t) = -\infty$. A non-Ponzi condition must be imposed

$$\lim_{t\to\infty} k(t)e^{-\int_0^t r(s)-nds} = 0 \quad (4)$$

to ensure that households cannot have exploding debt. This condition can be obtained from the integration of the linear first order differential equation (3). Further details and enlightening economic considerations can be obtained from and [1] and [2].
The problem is thus to maximize (1) restricted to (3) and (4):

$$\max_{k(t),c(t)} \int_0^{\infty} u(c(t))e^{-(\rho-n)t}dt \quad s.t.$$

$$\dot{k}(t) = (r(t) - n)k(t) + w(t) - \mathrm{c}(t) \quad (5)$$
$$k(0) > 0 \quad given$$
$$\lim_{t\to\infty} k(t)e^{-\int_0^t r(s)-nds} = 0 \quad (6)$$

This problem can be solved by the current-value of the Hamiltonian (see [3]):

$$\mathcal{H}(t, k(t), c(t), \lambda(t)) = u(c(t)) + \lambda(t)[(r(t) - n)k(t) + w(t) - \mathrm{c}(t)]$$

First order conditions give rise to

$$\frac{\partial \mathcal{H}}{\partial c}(t, k(t), c(t), \lambda(t)) = u'(c(t)) - \lambda(t) = 0 \quad (7)$$
$$\frac{\partial \mathcal{H}}{\partial k}(t, k(t), c(t), \lambda(t)) = \lambda(t)(r(t) - n) = (\rho - n)\lambda(t) - \dot{\lambda}(t) \quad (8)$$
$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda(t)k(t) = 0 \quad (9)$$

where (9) is the transversality condition. From (8) we obtain $\frac{\dot{\lambda}(t)}{\lambda(t)} = -(r(t) - \rho)$, and by differentiating (7), dividing by $\lambda(t)$ and relating to the previous expression, we obtain the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = (r(t) - \rho)\sigma(t) \quad (10)$$

where $\sigma(t) = -\frac{u'(c(t))}{u''(c(t))c(t)}$.

It can be shown that the non-Ponzi condition (4) is implied by the transversality condition (9). Indeed, noting that

$$\int \frac{\dot{\lambda}(t)}{\lambda(t)}dt = -\int r(t) - \rho \, dt$$

$$\log(\lambda(t)) + c_1 = -\int r(t) - \rho \, dt$$

$$\lambda(t) = c_2 e^{-\int r(t)-\rho dt}$$

one gets

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda(t)k(t) = \lim_{t\to\infty} k(t)e^{-\int_0^t r(s)-nds} = 0$$

2.2 Firms

Firms hire labor and rent capital to produce goods using a labor-augmenting (Harrod-neutral) technological change production function

$$Y(t) = F(K(t), A(t)L(t))$$

where the product (final good) $Y(t)$ is a function of capital $K(t)$ and effective labor $A(t)L(t)$, where $L(t)$ is the (homogeneous) labor supply and $A(t)$ an index of labor productivity (technological level). Assumptions on F must be taken: (i) concavity ($\frac{\partial F}{\partial K} > 0, \frac{\partial F}{\partial L} > 0, \frac{\partial^2 F}{\partial K^2} < 0, \frac{\partial^2 F}{\partial L^2} < 0$) and constant returns to scale (homogeneous function of degree one) and (ii) Inada conditions[4] to insure stability of the growth trajectory in neoclassic models. Consider that the technological is $(t) = e^{gt}A(0)$, $A(0) = 1$, where g is the long term exogenous growth rate, $g = \frac{\dot{A}(t)}{A(t)} \geq 0$.

Firms aim at maximizing profit

$$\max_{K,L} F(K, AL) - (r(t) + \delta)K(t) - w(t)L(t)$$

where δ is the depreciation rate on capital. First order conditions imply that $\frac{\partial F(.)}{\partial K} = r(t) + \delta$ and $\frac{\partial F(.)}{\partial L} = w(t)$. On the other hand, defining by $\hat{k}(t) = \frac{K(t)}{A(t)L(t)} = \frac{k(t)}{A(t)}$, the effective capital-labor ratio, $f(\hat{k}(t)) = \frac{F(.)}{A(t)L(t)}$, the production function in intensive form, one gets

$$\frac{\partial F(.)}{\partial K} = f'\left(\hat{k}(t)\right) = r(t) + \delta \quad (11)$$

$$\frac{\partial F(.)}{\partial L} = \left[f\left(\hat{k}(t)\right) - f'\left(\hat{k}(t)\right)\hat{k}(t)\right]A(t) = w(t) \quad (12)$$

Writing (5) in intensive form, bearing in mind $\hat{c}(t) = \frac{c(t)}{A(t)}$ and $\dot{k}(t) = \frac{d\left(\hat{k}(t)\right)A(t))}{dt}$,

$$\frac{\dot{k}(t)}{A(t)} = (r(t) - n)\hat{k}(t) + \frac{w(t)}{A(t)} - \hat{c}(t)$$

$$\dot{\hat{k}}(t) + \hat{k}(t)g = (r(t) - n)\hat{k}(t) + \frac{w(t)}{A(t)} - \hat{c}(t)$$

and considering (11)-(12) one gets the intertemporal budget constraint in intensive form

$$\dot{\hat{k}}(t) = f\left(\hat{k}(t)\right) - (\delta + n + g)\hat{k}(t) - \hat{c}(t); \quad (13)$$

also, the non-Ponzi condition writes

$$\lim_{t\to\infty} \hat{k}(t)e^{-\int_0^t f'\left(\hat{k}(t)\right)-\delta-n-gds} = 0. \quad (14)$$

Moreover, the Euler equation can be written as

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{\dot{c}(t)}{c(t)} - g = (r(t) - \rho)\sigma(t) - g \quad (15)$$

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \left(f'\left(\hat{k}(t)\right) - \delta - \rho - \frac{g}{\sigma(t)}\right)\sigma(t). \quad (16)$$

Note that $\frac{\dot{c}(t)}{c(t)} = \left(\frac{\partial f(k)}{\partial k} - (\delta + \rho)\right)\sigma(t)$.

The system of differential equations to be solved is thus formed by equations (13) and (16).

The model presented is based on a decentralized economy. It could have been developed as a centralized economy, looking for the solution for the problem of a social planner.

2.3 Connection with the Solow-Swan model

First note that taking the propensity to consume as constant, $1 - s$, and $\hat{c}(t) = (1 - s)f(\hat{k}(t))$, from (13) one recovers $\dot{\hat{k}}(t) = sf\left(\hat{k}(t)\right) - (\delta + n + g)\hat{k}(t)$, which is the movement equation for the Solow-Swan model.

2.4 Specifying production and utility functions

Let us consider the Cobb-Douglas production function, $f(k(t)) = k(t)^\alpha$, where α is the capital share in production, and the Constant Intertemporal Elasticity of Substitution, CIES, utility function, $u(c(t)) = \frac{c(t)^{1-\theta}}{1-\theta}$, $\theta \neq 1$. Both functions agree with the required assumptions. The latter provides a degree to which people prefer a stable rate of consumption relative to higher consumption in the future; it is also referred as Constant Relative Risk Aversion, CRRA, since it assigns a constant ratio by which people give higher weights to downside risks than to upside ones. Also, it is easy to verify that $\sigma = \frac{1}{\theta}$:

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta} = \frac{\alpha k(t)^{\alpha-1} - \delta - \rho}{\theta} \quad (17)$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'\left(\hat{k}(t)\right) - \delta - \rho - \theta g}{\theta} = \frac{\alpha \hat{k}(t)^{\alpha-1} - \delta - \rho - \theta g}{\theta} \quad (18)$$

2.5 Steady state

The steady state equilibrium is an equilibrium path in which capital-labor ratio, consumption and output are constant (($\dot{\hat{\boldsymbol{c}}}(\boldsymbol{t}) = \dot{\hat{\boldsymbol{k}}}(\boldsymbol{t}) = \boldsymbol{0}$)

4

$$\lim_{K\to 0} \frac{\partial F}{\partial K} = \lim_{L\to 0} \frac{\partial F}{\partial L} = \infty \; and \; \lim_{K\to\infty} \frac{\partial F}{\partial K} = \lim_{L\to\infty} \frac{\partial F}{\partial L}$$

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$$\begin{cases} f\left(\hat{k}(t)\right) - (\delta + n + g)\hat{k}(t) - \hat{c}(t) = 0 \\ f'\left(\hat{k}(t)\right) - \delta - \rho - \theta g = 0 \end{cases}$$

and, considering the Cobb-Douglas and CIES functions, one gets

$$\begin{cases} \hat{k}^* = \left(\frac{\alpha}{\delta + n + \theta g}\right)^{\frac{1}{1-\alpha}} \\ \hat{c}^* = (\hat{k}^*)^\alpha - (\delta + n + g)\hat{k}^* \end{cases} \quad (19)$$

2.6 Transition dynamics

The sought solution $(\hat{c}^*, \hat{k}^*)$ is a saddle point, and to access this let us examine the local properties, linearizing. Taylor expansion of first order gives rise to

$$\begin{bmatrix} \dot{\hat{c}}(t) \\ \dot{\hat{k}}(t) \end{bmatrix} \approx J_{|(\hat{c}^*,\hat{k}^*)} \begin{bmatrix} \hat{c} - \hat{c}^* \\ \hat{k} - \hat{k}^* \end{bmatrix}$$

where the Jacobian at the equilibrium point is

$$J_{|(\hat{c}^*,\hat{k}^*)} = \begin{bmatrix} \frac{\partial \dot{\hat{c}}(t)}{\partial \hat{c}} & \frac{\partial \dot{\hat{c}}(t)}{\partial \hat{k}} \\ \frac{\partial \dot{\hat{k}}(t)}{\partial \hat{c}} & \frac{\partial \dot{\hat{k}}(t)}{\partial \hat{k}} \end{bmatrix}_{|(\hat{c}^*,\hat{k}^*)}$$

$$= \begin{bmatrix} \frac{f'\left(\hat{k}(t)\right) - (\delta + \rho + \theta g)}{\theta} & \frac{f''\left(\hat{k}(t)\right)\hat{c}(t)}{\theta} \\ -1 & f'\left(\hat{k}(t)\right) - (\delta + n + g) \end{bmatrix}_{|(\hat{c}^*,\hat{k}^*)}$$

$$= \begin{bmatrix} 0 & \frac{f''(\hat{k}^*)\hat{c}^*}{\theta} \\ -1 & \rho - n - (1-\theta)g \end{bmatrix}$$

since in equilibrium $f'\left(\hat{k}(t)\right) - (\delta + \rho + \theta g) = 0$. The characteristic polynomial is thus $\xi^2 - (\rho - n - (1-\theta)g)\xi + \frac{1}{\theta} f''(\hat{k}^*)\hat{c}^* = 0$. Since $\rho - n - (1-\theta)g > 0$ and $f''(\hat{k}^*)\hat{c}^* < 0$, there are two real eigenvalues, one positive and another negative. The equilibrium point is a saddle point.

3. Deduction of the model: symbolic computation

```
clear all; clc;
disp('----------------------------------');
disp('   Ramsey-Cass-Koopmans model:    ');
disp('deduction (symbolic computations)' );
disp('----------------------------------');
```

```
---------------------------------
   Ramsey-Cass-Koopmans model:
deduction (symbolic computations)
---------------------------------
```

1. Households

```
syms t L(t) K(t) C(t) a(t) r(t) w(t) u(t) rho n
% variables
L(t) = exp(n.*t);        % total population at t
n = diff(L(t),t)/L(t);   % labor growth rate
c(t) = C(t)/L(t);        % per capita consumption
k(t) = K(t)/L(t);        % per capita capital
```

... lifetime utility
each household as a lifetime utility given by

```
pretty(simplify(int(u(c)*exp(-rho*t)*L(t),0,inf)));
```

```
  Inf
    /
   |
   |  u(exp(-n t) C(t)) exp(t (n - rho)) dt
  /
   0
```

... budget constraint

the households budget constraint is given by

```
eq = diff(a)== (r(t)-n)*a(t)+w(t)-c(t);
% per capita asset accumulation
% r(t) risk-free market flow rate
% w(t) labor income earnings
pretty(eq);
```

```
D(a)(t) == w(t) - a(t) (n - r(t)) - exp(-n t)
C(t)
```

... non-Ponzi game condition

```
syms s %a(t) r(s) t n w(t) c(t)
pretty(limit(a(t)*exp(-(int((r(s)-n),0,t))),t,inf)>=0);
```

```
          /   / t                \              \
          |   | /                |              |
          |   | |                |              |
0 <= limit|exp|-| r(s) - n ds|a(t), t = Inf|
          |   | /                |              |
          \   \ 0                /              /
```

that is

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(s)-n ds} \geq 0$$

solve the linear diff equation: budget constraint

pretty(simple(dsolve(diff(a) == (r(t)-n)*a(t)+w(t)-c(t))))

```
   / /                \/      /   /  /            \
   | |                ||      |   |  |            |
exp| | r(t) dt - n t||C2 - |exp|- |   r(t) dt|
   \ /                /\      /   \  /            /

                                       \
                                       |
   (C(t) - exp(n t) w(t)) dt |
                                       /
```

taking $\lim_{t\to\infty}$ and considering the non-Ponzi game condition, we reach
$\int_0^\infty c(t)e^{-\int_0^t r(s)-nds}\,dt \le \int_0^\infty w(t)e^{-\int_0^t r(s)-nds}$dt +a(0)
that is, the present value of consumption must not exceed total wealth.

... Hamiltonian and FOC

syms beta theta rho t r k n g w c u(c) lambda lambdadot

utility function

```
u(c) = c^(1-theta)/(1-theta);
```

t at equilibrium, a(t) = k(t): rate of change of assets = capital per capita

Hamiltonian

```
H = u(c)+lambda*((r-n)*k+w-c);
% prepare variables as function of t
ct = sym('c(t)'); kt = sym('k(t)');
rt = sym('r(t)'); wt = sym('w(t)');
lambdat = sym('lambda(t)');
```

first order conditions: $\frac{\partial H}{\partial c} = 0$

```
dHdc = diff(H,c);
eq1 = dHdc == 0; pretty(eq1);
```

```
       1
    ------ - lambda == 0
    theta
  c
```

solve with respect to λ

```
sol1 = simple(solve(eq1,lambda));
sol1 = subs(sol1,{c,k,r,lambda},…
       {ct,kt,rt,lambdat});
pretty(sol1); % sol1 = lambda
```

```
       1
   ---------
     theta
 c(t)
```

first order conditions: $\frac{\partial H}{\partial k} = (\rho - n) * \lambda - \dot{\lambda}$

```
dHdk = diff(H,k);
dHdk = subs(dHdk,lambda,lambdat);
eq2 = dHdk == (rho-n)*lambdat-diff(lambdat,t);
pretty(eq2);
eq2 = subs(eq2,'diff(lambda(t), t)',lambdadot);
```

```
-lambda(t)  (n  -  r)  ==  -  diff(lambda(t),  t)  -
lambda(t)  (n - rho)
```

solve with respect to $\dot{\lambda}$

```
sol2 = simple(solve(eq2,lambdadot));
pretty(sol2); % sol2 = lambdadot
```

```
-lambda(t)  (r - rho)
```

$$\frac{\dot{\lambda}}{\lambda}$$

```
pretty(sol2/lambdat);
```

```
  rho - r
```

differentiation with respect to time is required:

```
dHdct =
subs(dHdc,{c,k,r,lambda},{ct,kt,rt,lambdat});
eq3  = diff(dHdct==0,t);
eq3  = subs(eq3,'diff(lambda(t), t)',lambdadot);
sol3 = simple(solve(eq3,lambdadot));
sol3 = simple(sol3/sol1); pretty(sol3);
```

```
    theta diff(c(t), t)
  - -------------------
           c(t)
```

... Euler equation

```
eq4 = sol3 == sol2/lambdat; pretty(eq4);
```

```
    theta diff(c(t), t)
  - ------------------- == rho - r
           c(t)
```

2. Firms

```
syms t L K A alpha w delta r g kh
```

... production function maximization

```
% production function
F = K^alpha*(L*A)^(1-alpha);
```

```
% function to optimize
profit = F-(r+delta)*K-w*L;
```

interest rate

```
r = subs(solve(diff(profit,K),r),...
    'K^(alpha - 1)*alpha*(A*L)^(1 - alpha)',...
    'alpha*kh^(alpha-1)');
pretty(r);
```

```
          alpha - 1
  alpha kh          - delta
```

wage

```
w = subs(solve(diff(profit,L),w),…
    1/(A*L)^alpha,kh^alpha/K^alpha);
w = subs(w,'A',exp(g*t)); pretty(w);
```

```
      alpha
  - kh      exp(g t) (alpha - 1)
```

... movement equation in effective terms

```
syms  wt rt n k(t) ktil kdot khdot c ch cdot
L = exp(n*t);    % total population at t
A = exp(g*t);    % total tech level at t
kh = k(t)/A;     % capital in effective terms
% diff. mov. eq.
eq5 = diff(k(t),t)==(rt-n)*k(t)+wt-c;
% diff. khdot
eq6 = khdot==diff(kh);
% solve for kdot
kdot = solve(subs(eq6,'diff(k(t),t)',kdot),kdot);
% equate eq.'s 5 and 6
eq7 = subs(eq5,'diff(k(t), t)',kdot);
khdot = expand(solve(eq7,khdot));
khdot = subs(khdot,'- c*exp(-g*t)','-ch');
khdot = subs(khdot,'- g*exp(-g*t)*k(t)','-g*kh');
khdot = subs(khdot,'- n*exp(-g*t)*k(t)','-n*kh');
khdot = subs(khdot,'+ rt*exp(-
g*t)*k(t)','+rt*kh');
% plug expressions for r and w
khdot = expand(subs(khdot,{rt,wt},{r,w}));
khdot = collect(khdot,'kh'); pretty(khdot);
```

```
                             alpha
  (- delta - g - n) kh + kh       - ch
```

This is one differential equation (mov. equation).

... Euler eq. in effective terms

```
syms chdot c(t)
eq4 = subs(eq4,'diff(c(t), t)',cdot);
eq4 = subs(eq4,'c(t)',c);
eq4 = collect(subs(eq4,'r',r),theta);
ch = c(t)/A;
eq8 = chdot==diff(ch);
cdot = solve(subs(eq8,'diff(c(t),t)',cdot),cdot);
% plug cdot function of chdot in eq. 4
eq9 = subs(eq4,'cdot',cdot);
cc = A*ch; eq9 = subs(eq9,'c(t)',cc);
chdot = solve(eq9,chdot);
pretty(chdot/ch);
```

```
                                alpha - 1
    delta + rho + g theta - alpha kh
  - ----------------------------------------
                     theta
```

This is the other differential equation (Euler equation).

4. Solution of the model: numerical computation

```
disp('-----------------------------------');
disp('   Ramsey-Cass-Koopmans model:     ');
disp('simulation (numerical computations)');
disp('-----------------------------------');
```

```
-----------------------------------
    Ramsey-Cass-Koopmans model
simulation (numerical computations)
-----------------------------------
```

```
global alpha delta rho n g theta kss css k0
alpha  =  0.3;    %  elasticity  of  capital  in
production
delta = 0.05; % depreciaton rate
rho   = 0.02; % time preference
n     = 0.01; % population growth
g           =  0.00;  %  exogenous  growth  rate  of
technology
theta = (delta+rho)/(alpha*(delta+n+g)-g);
% inverse intertemporal elasticity of
% substitution;
%  select  theta  so  that  the  saving  rate  is
constant
% s=1/theta
```

Steady state values and shock

```
kss = ((delta+rho+g*theta)/alpha)^(1/(alpha-1));
css = kss^alpha-(n+delta+g)*kss;
k0  = 0.1*kss;     % shock at k
disp('steady-state:')
fprintf('k* = %14.6f \n',kss);
fprintf('c* = %14.6f \n',css);
fprintf('shock - initial value for k: k0 = %14.6f
\n',k0);
```

```
steady-state:
k* = 7.996323
c* = 1.386029
shock - initial value for k: k0 = 0.799632
```

Exact solution

```
for this model an analytical solution is known
k = @(t) (1./((delta+n+g).*theta)+(k0.^(1-alpha)-
...
  1./((delta+n+g).*theta))*exp(-(1-alpha)...
  .*(delta+n+g).*t)).^(1./(1-alpha));
c = @(t) (1-1./theta).*k(t).^alpha;
```

Approximate solution using bvp4c (matlab solver)

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the RCK model is a BVP problem

```
nn      = 100;
solinit = bvpinit(linspace(0,nn,5),[0.5 0.5]);
sol     = bvp4c(@ode_bvp,@bcs,solinit);
xint    = linspace(0,nn,50);
Sxint   = deval(sol,xint);
```

Plot both the analytical and numerical solution from bvp4c}

```
subplot(2,1,1); hold on
% analytical and approx. sol. for k
ezplot(k,[0,nn]);
plot(xint,Sxint(1,:),'r.');
plot(0,k0,'go');
legend('exact','bvp4c','$k_0$ initial value',...
       'location','SouthEast');
set(legend,'Interpreter','latex');
xlabel('$t$','Interpreter','LaTex');
ylabel('$k$','Interpreter','LaTex');
subplot(2,1,2); hold on
% analytical and approx. sol. for c
ezplot(c,[0,nn])
plot(xint,Sxint(2,:),'r.')
legend('exact','bvp4c','location','SouthEast');
set(legend,'Interpreter','latex');
xlabel('$t$','Interpreter','LaTex');
ylabel('$c$','Interpreter','LaTex');
```

Functions:

```
% Boundary conditions
function res = bcs(ya,yb)
% Y=[k(t); c(t)]
% ya and yb for initial and final conditions
global kss k0
res = [ya(1)-k0; yb(1)-kss];

% Differential system speciification
function dydt=ode_bvp(~,y)
% Y=[k(t); c(t)]
global alpha delta rho n g theta
dydt = [...
  (y(1)^alpha-y(2)-(n+delta+g)*y(1)); ...
  (y(2)*(alpha*y(1)^(alpha-1)-...
    (delta+rho+g*theta))/theta);
      ]
```

Fig. 1 shows the approximate solution against the analytical one (known for this particular case). The approximation agrees perfectly with the expected solution.

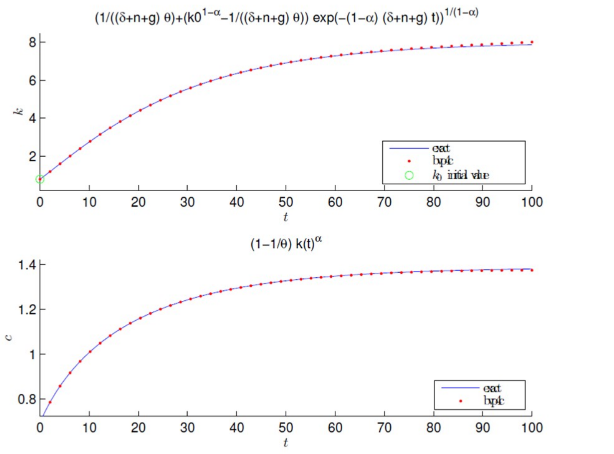

Fig. 1 RCK transition dynamics: analytical solution (line) and numerical approximation (dots).

Other graphical figures can also be explored, like the phase diagram drawn in Fig. 2.

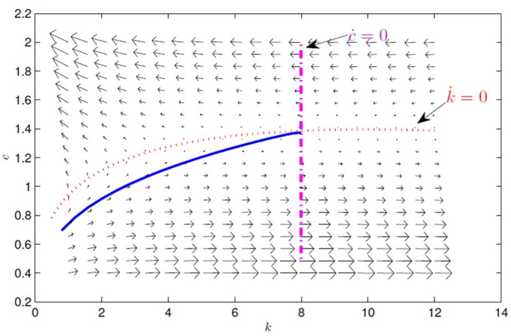

Fig. 2 RCK phase diagram.

Shocks on the economy can be easily performed and the numerical results comfortably obtained.

5. Conclusions

This work provides an integrated MATLAB approach to tackle typical economic growth models, namely, infinite-horizon optimal-control problems. We have illustrated, based on a neoclassical growth cornerstone model in economic theory, that (i) symbolic computations can be used to deduce the system of differential equations governing the model, (ii) powerful methods of numerical computation embedded in MATLAB can be employed to solve the associated boundary value problem and (iii) visualization procedures provided by the interactive environment allow for enlightening economic interpretation of the results. Being able to perform all these

steps on top of the same software library is of great help to develop and solve problems, in particular those that are complex either for the problem itself or for the large number of equations involved.
The symbolic part could not be done with ease. Certainly, other packages, just intended for symbolic computations allow for a smoother development. Instead, the aim was to explore one single package, the one delivering the best numerical performance.

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