ACSIJ Advances in Computer Science: an International Journal, Vol. 3, Issue 5, No.11 , September 2014
ISSN : 2322-5157
www.ACSIJ.org

Taylor Approach for Solving Non-Linear Bi-level Programming Problem

Eghbal Hosseini[1], Isa Nakhai Kamalabadi[2]

[1] Department of Mathematics, Payamenur University of Tehran, Tehran, Iran
eghbal_math@yahoo.com

[2] Department of Industrial Engineering, University of Kurdistan, Sanandaj. Iran
nakhai.isa@gmail.com

Abstract

In recent years the bi-level programming problem (BLPP) is interested by many researchers and it is known as an appropriate tool to solve the real problems in several areas such as computer science, engineering, economic, traffic, finance, management and so on. Also it has been proved that the general BLPP is an NP-hard problem. The literature shows a few attempts for using approximate methods. In this paper we attempt to develop an effective approach based on Taylor theorem to obtain an approximate solution for the non-linear BLPP. In this approach using the Karush-Kuhn–Tucker, the BLPP has been converted to a non-smooth single problem, and then it is smoothed by the Fischer – Burmeister function. Finally the smoothed problem is solved using an approach based on Taylor theorem. The presented approach achieves an efficient and feasible solution in an appropriate time which has been is evaluated by comparing to references and test problems.

Keywords: *Non-linear bi-level programming problem, Taylor theorem, Karush-Kuhn–Tucker conditions, smoothing methods.*

1. Introduction

The bi-level programming problem (BLPP) is a nested optimization problem, which has two levels in hierarchy. The first level is called leader and the second one is called follower. They have their own objective functions and constraints. The leader actions first, and the follower reacts to the leader decision. The follower should optimize its objective function according to the leader decision and delivered answers of the leader. In fact, the leader inflicts his decision on and obtains reaction of the follower.

It has been proved that the BLPP is an NP- Hard problem even to seek for the locally optimal solutions [1, 2]. Nonetheless the BLPP is an applicable problem and a practical tool to solve decision making problems. It is used in several areas such as transportation, finance and so on. Therefore finding the optimal solution has a special importance to researchers.

Several algorithms have been presented for solving the BLPP [3, 4, 11, 12, 13, 21, 25]. These algorithms are divided into the following classes: Transformation methods [3, 4, 22, 23, 36], Fuzzy methods [5, 6, 7, 8, 24, 35], Global techniques [9, 10, 11, 12, 38, 39], Primal–dual interior methods [13], Enumeration methods [14], Meta heuristic approaches [15, 16, 17, 18, 19, 25, 37, 40, 41].

The purpose of this paper is to develop two efficient approaches for solving linear bi-level programming problems (LBPP). We mainly concentrate on LBPP, in which both the upper level objective function and the lower level objective function are convex functions. In the present work, first, different from all previous works, we use a new proposed function to smoothen the problem. Then, an approximate approach is proposed which provides an efficient solution requiring much less times as compared to already available methods. The remainder of the paper is structured as follows: in Section 2, basic concepts of the non-linear BLPP and a smooth method to BLPP are introduced. Main theoretical results and steps of proposed algorithm are presented in Section 3. Computational results are presented for in Section 4. Finally, the paper is finished in Section 5 by presenting the concluding remarks.

2. Non-Linear BLPP and Smoothing Method

The BLPP is used frequently by problems with decentralized planning structure. It is defined as [20]:

$$\min_{x} F(x, y)$$

$$s.t \min_{y} f(x, y) \qquad (1)$$

$s.t\ g(x,y) \le 0,$

$x, y \ge 0.$

Where

$F: R^{n\times m} \to R^1, f: R^{n\times m} \to R^1,$

$g: R^{n\times m} \to R^q, x \in R^n, y \in R^m.$

Also F and f are objective functions of the leader and follower respectively.
The feasible region of the non-linear BLP problem is
$S = \{(x,y) | g(x,y) \le 0, x, y \ge 0\}$ (2)

Using KKT conditions problem (1) can be converted into the following problem:

$$\min_{x,y\mu} F(x,y,\mu)$$
$$s.t\ \nabla_y L(x,y,\mu) = 0,$$
$$\mu g(x,y) = 0, \quad (3)$$
$$g(x,y) \le 0,$$
$$\mu \ge 0.$$

Where L is the Lagrange function and $L(x,y,\mu) = f(x,y) + \mu g(x,y).$

Because problem (3) has a complementary constraint, it is not convex and it is not differentiable. Fortunately Facchinei et al, 1999 proposed smooth method for solving problem with complementary constraints and we use this method to smooth problem (3).

In general the BLPP is a non-convex optimization problem therefore there is no general algorithm to solve it. This problem can be non-convex even when all functions and constraints are bounded and continuous.

Definition 2.1:

Fischer-Burmeister is the following function,

$\phi: R^2 \to R, \phi(m,n) = m + n - \sqrt{m^2+n^2}$ or $\phi: R^3 \to R$, $\phi(m,n,\varepsilon) = m + n - \sqrt{m^2+n^2+\varepsilon},$ where $m \ge 0$, $n \ge 0$, $\phi(m,n) = 0 \Leftrightarrow mn = 0, m \ge 0, n \ge 0.$

Using the Fischer-Burmeister function $\phi(m,n,\varepsilon) = m + n - \sqrt{m^2+n^2+\varepsilon}$ in problem (3), we obtain the following problem:

$$\min F(x,y,\mu)$$
$$s.t\ \nabla_y L(x,y,\mu) = 0,$$
$$\mu_i - g_i(x,y) - \sqrt{\mu_i^2 + g_i^2(x,y) + \varepsilon} = 0, i = 1,2,\ldots,m, \quad (4)$$
$$x, y, \mu_i \ge 0, i = 1,\ldots,m.$$

Which $m = \mu_i \ge 0, n = -g_i(x,y) \ge 0,$ and $g_i(x,y), \mu_i, a^i, b^i$ are i-th row of g, μ, A, B respectively.

Let:

$$G(x,y,\mu) = \begin{bmatrix} \mu_1 - g_1(x,y) - \sqrt{\mu_1^2 + g_1^2(x,y) + \varepsilon} \\ \mu_2 - g_2(x,y) - \sqrt{\mu_2^2 + g_2^2(x,y) + \varepsilon} \\ \vdots \\ \mu_m - g_m(x,y) - \sqrt{\mu_m^2 + g_m^2(x,y) + \varepsilon} \end{bmatrix}, \quad (5)$$

$H(x,y,\mu) = \nabla_y L(x,y,\mu).$

Problem (4) can be written as follows,

$$\min F(x,y,\mu)$$
$$s.t\ H(x,y,\mu) = 0,$$
$$G(x,y,\mu) = 0, \quad (6)$$
$$x, y, \mu \ge 0.$$

Where $t = (x,y,\mu) \in R^{n+2m}$

By applying Taylor theorem at a feasible point such as t^k for function G, H, F and take only two linear part of them, the following linear functions is constructed:

$$G_i(t^k) + \nabla G_i(t^k)(t - t^k) = 0, \quad i = 1,2,\ldots m.$$
$$H_i(t^k) + \nabla H_i(t^k)(t - t^k) = 0, \quad i = 1,2,\ldots m \quad (7)$$
$$F_i(t^k) + \nabla F_i(t^k)(t - t^k) = 0, \quad i = 1,2,\ldots m$$

Because the obtained problem by using Taylor theorem is linear programming, it can be solved using simplex methods.

A summary of important properties for convex problem as follows, which $f: S \to R^n$ and S is a nonempty convex set in R^n.

(1) The convex function f is continuous on the interior of S.
(2) Every local optimal solution of *f* over a convex set $X \subseteq S$ is the unique global optimal solution.

ACSIJ Advances in Computer Science: an International Journal, Vol. 3, Issue 5, No.11 , September 2014
ISSN : 2322-5157
www.ACSIJ.org

(3) If $\nabla f(\bar{x}) = 0$, then $\bar{x}$ is unique global optimal solution of f over S.

Since in problem (3), most of the equality constraints are not linear then it concerns that the above problem is a non-convex programming problem, which indicates there are local optimal solutions that are not global solutions. Therefore solving the problem (3) will be complicated and we use the following method for solving this problem.

3. Main theoretical results and steps of algorithm

Definition 3.2: A metric space is pair (X,d) where X is a set and d is a metric on X and:

(i) $d \geq 0$,
(ii) $d(x,y) = 0 \Leftrightarrow x = y$,
(iii) $d(x,y) = d(y,x)$,
(iv) $d(x,y) \leq d(x,z) + d(z,y)$.

Definition 3.4: A sequence $\{x_n\}$ is said to Cauchy if for every $\varepsilon > 0$ there is an N such that

$$\forall_{m>r>N} \quad |x_m - x_r| < \varepsilon.$$

Theorem 3.1: All polynomials continuous everywhere. Additionallyx^n, $\sqrt[n]{x}$ are continuous for all x., when n is odd and for x>0, when n is even.

Proof:

The proof of this theorem has been proposed in [30].

Theorem 3.2: Suppose that f and g are continuous at x=a. Then $f+g$, $f-g$ are continuous at x=a.

Proof:

The proof has been given by [30].

Theorem 3.3: Suppose that $\lim_{x\to a} g(x) = L$ and f is continuous at L. Then,

$$\lim f(_{x\to a} g(x)) = f(\lim_{x\to a} g(x)) = f(L)$$

The proof has been given by [30].

Corollary 3.1: Suppose that g is continuous at a and f is continuous at g(a). Then, the composition fog is continuous at a.

Proof:

From above theorem, we have:

$$\lim_{x\to a}(fog)(x) = \lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(g(a)) = (fog)(a).$$

This finished the proof.

Because functions G, H in (6) is always continuous everywhere and it is possible to use Theorems and corollary, Taylor Theorem for it in (6) and F should be continuous too.

Theorem 3.4 (Taylor Theorem) [30]: Suppose f has n +1 continuous derivatives on an open interval containing a. Then for each x in the interval,

$$f(x) = \left[\sum_{k=0}^{n} \frac{f^{k}(a)}{k!}(x-a)^k\right] + R_{n+1}(x)$$

Proof:

The proof of this theorem was given by [30].

According above theorems and definitions F, f, g, H, G are continuous and differentiable, also F, f are convex. We mention that these conditions are necessary for proposed approach in this paper.

Steps of the proposed algorithm as follows:

Step 1: Initialization

The feasible point t^1 is created randomly, error ε_1 is given and suppose k=1.

ε_1 is a small and appropriate given error and finishing the algorithm depends to ε_1 such that it is finished whenever difference between produced solutions by the algorithm in two consecutive iterations is less than ε_1.

Step 2: finding solution.

Using Taylor theorem for $G(t), H(t)$ *and* $F(t)$ at t^k, we obtain following problem:

$$\begin{aligned} \min \quad & F_i(t^k) + \nabla F_i(t^k)(t - t^k) \\ \text{s.t} \quad & H_i(t^k) + \nabla H_i(t^k)(t - t^k) = 0, \quad i = 1,2,\dots m, \\ & G_i(t^k) + \nabla G_i(t^k)(t - t^k) = 0, \quad i = 1,2,\dots m, \\ & x, y, \mu_i \geq 0, i = 1, \dots, m. \end{aligned} \tag{8}$$

Solve the problem (8) using simplex method (by MATLAB 7.1). By solving this problem, an optimal solution such as t^{k+1} is obtained for (8).

Step 3: Keeping the present best solution.

Because (8) is an approximation for (6) by Taylor theorem, therefore optimal solution for (11) is an approximation of optimal solution for (6). Thus t^{k+1} can be a good approximation of problem (6) optimal solution. Therefore let $t^* = t^{k+1}$ and go to next step.

Step 4: Termination

If $d(F(t^{k+1}), F(t^k)) < \varepsilon_1$ then the algorithm is finished and t^* is the best solution by the proposed algorithm. Otherwise, let k=k+1 and go to the step 2. Which d is metric and,

$$d\left(F(t^{k+1}), F(t^k)\right) = \left(\sum_{i=1}^{n+2m} (F(t_i^{k+1}) - F(t_i^k))^2\right)^{\frac{1}{2}}.$$

Theorem 3.5: Every Cauchy sequence in real line and complex plan is convergent.

Proof:

Proof of this theorem is given in [34].

Theorem 3.6: Sequence $\{F_k\}$ which was proposed in above algorithm is convergent to the optimal solution, so that the algorithm is convergent.

Proof

Let $(F_l) = (F(t^l)) = (F(t_1^l), F(t_2^l), \dots, F(t_{n+2m}^l))$
$= (F_1^{(l)}, F_2^{(l)}, \dots, F_{n+2m}^{(l)})$.

According to step 4

$$d(F_{k+1}, F_k) = d\left(F(t^{k+1}), F(t^k)\right)$$

$$= \left(\sum_{i=1}^{n+2m} (F(t_i^{k+1}) - F(t_i^k))^2\right)^{\frac{1}{2}} < \varepsilon_1$$

Therefore, $\left(\sum_{i=1}^{n+2m}\left(F\left(t_i^{k+1}\right)-F\left(t_i^{k}\right)\right)^2\right) < \varepsilon_1{}^2$. There is large number such as N which k+1>k>N and j=1,2,…,2m+n we have:
$(F_j^{(k+1)} - F_j^{(k)})^2 < \varepsilon_1{}^2$, therefore $|F_j^{(k+1)} - F_j^{(k)}| < \varepsilon_1$
Now let m=k+1, r=k then we have
$\forall_{m>r>N} \quad |F_j^{(m)} - F_j^{(r)}| < \varepsilon_1$.
This shows that for each fixed j, (1≤j≤2m+n), the sequence $(F_j^{(1)}, F_j^{(2)}, \dots)$ is a Cauchy sequence of real numbers, then it converges by theorem 3.5.
Say, $F_j^{(m)} \to F_j$ as $m \to \infty$. Using these 2m+n limits, we define $F = (F_1, F_2, \dots, F_{2m+n})$. From step 4 and m=k+1, r=k,
$d(F_m, F_r) < \varepsilon_1$
Now if $r \to \infty$, then $d(F_m, F) \le \varepsilon_1$.
This shows that F is the limit of (F_m) and the sequence is convergent by definition 3.3 therefore proof of theorem is finished.

4. Computational results

Example 1[31]:

Consider the following non-linear bi-level programming problem:

$$\begin{aligned} &\min_x \; x^2 + (y-10)^2 \\ &s.t \; \min_{y\ge 0} \; (x+2y-30)^2 \\ &\qquad s.t \; x - y^2 \ge 0, \\ &\qquad\quad 20 - x - y^2 \ge 0 \\ &\qquad\quad 0 \le x \le 15. \end{aligned}$$

Using KKT conditions the following problem is obtained:

$$\begin{aligned} &\min_x \; x^2 + (y-10)^2 \\ &s.t \; 4(x+2y-30) = 0, \\ &\quad 2y(\lambda_1+\lambda_2) = 0, \\ &\quad \lambda_1(y^2 - x) = 0, \\ &\quad \lambda_2(y^2 + x - 20) = 0, \\ &\quad \lambda_3(x-15) = 0 \\ &\quad y^2 - x \le 0, \\ &\quad y^2 + x - 20 \le 0, \\ &\quad x - 15 \le 0, \\ &\quad \lambda_1, \lambda_2, \lambda_3 \ge 0. \end{aligned}$$

By the Fischer – Burmeister function, the above problem as follows:

$$\begin{aligned} &\min \; x^2 + (y-10)^2 \\ &s.t \; 4(x+2y-30) = 0, \\ &\quad (\lambda_1+\lambda_2) - 2y - \sqrt{(\lambda_1+\lambda_2)^2 + (2y)^2 + \varepsilon} = 0, \\ &\quad \lambda_1 - (y^2 - x) - \sqrt{\lambda_1^2 + (y^2-x)^2 + \varepsilon} = 0, \\ &\quad \lambda_2 - (y^2 + x - 20) - \sqrt{\lambda_2^2 + (x+y^2-20)^2 + \varepsilon} = 0, \\ &\quad \lambda_3 - (x-15) - \sqrt{\lambda_3^2 + (x-15)^2 + \varepsilon} = 0, \end{aligned}$$

We solve this problem using the proposed line search algorithm and we present the optimal solution in Table 1. By solving this problem the best solutions are found according to Table 1. It declares that the best solutions by the proposed algorithm are better than the best solution by the references in less time.

Table 1 comparison optimal solutions - Example 1

Best solution by our method ε=0.001		***Best solution according to reference [30]***		***Optimal solution***	
(x^*, y^*)	z^*	(x^*, y^*)	z^*	(x^*, y^*)	z^*
(2.6,1.612)	-77.11	(2.6,1.612)	-77.10	(2.6,1.612)	-77.11

Behavior of the variables in Example 1 has been show in figure 1 that variables x and y will be stable after 5000 and 4850 iterations respectively.

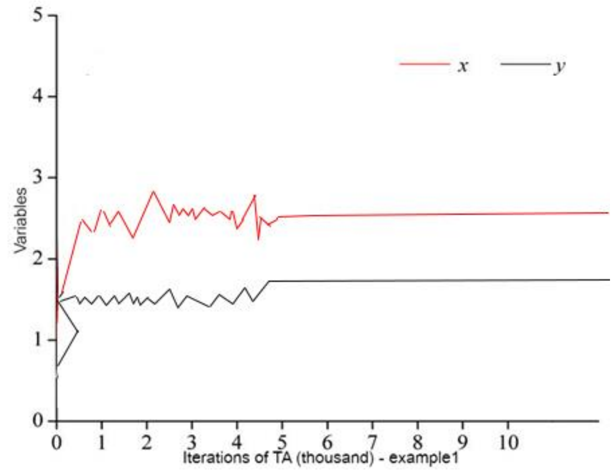

Figure 1. The transient behavior of the variables in Example 1

Example 2[4]:

Consider the following linear bi-level programming problem.

$$\min_{x} \; -x_1^2 - 2x_1 + x_2^2 - 2x_2 + y_1^2 + y_2^2$$
$$s.t \quad \min \; y_1^2 - 2x_1 y_1 + y_2^2 - 2x_2 y_2$$
$$s.t \quad .25 - (y_1 - 1)^2 \geq 0,$$
$$.25 - (y_2 - 1)^2 \geq 0.$$

After applying KKT conditions and smoothing method, and then proposed penalty function above problem will be transformed to the following problem:

$$\begin{aligned} \min_{x} \; & -x_1^2 - 2x_1 + x_2^2 - 2x_2 + y_1^2 + y_2^2 \\ s.t \; & + \mu_1 (2y_1 - 2x_1 + 2y_2 - 2x_2) \\ & + \mu_2 (\lambda_1 - (y_1 - 1)^2 + .25 - \sqrt{\lambda_1^2 + ((y_1 - 1)^2 + .25)^2 + \varepsilon} \;)^2 \\ & + \mu_3 (\lambda_2 + .25 - (y_2 - 1)^2 - \sqrt{\lambda_2^2 + ((y_2 - 1)^2 + .25)^2 + \varepsilon} \;)^2 \end{aligned}$$

Table 2 comparison optimal solutions - Example 2

Best solution by our method ε=0.001		***Optimal solution***	
(x^*, y_1^*, y_2^*)	z^*	(x^*, y_1^*, y_2^*)	z^*
(0.51,0.51,0.51,0.51)	-1.598	(0.51,0.51,0.51,0.51)	-1.598

The optimal solution is obtained using our method according to Table 2. Behavior of the variables in Example 2 has been show in figure 2 that variables will be stable after 6 thousand iterations respectively.

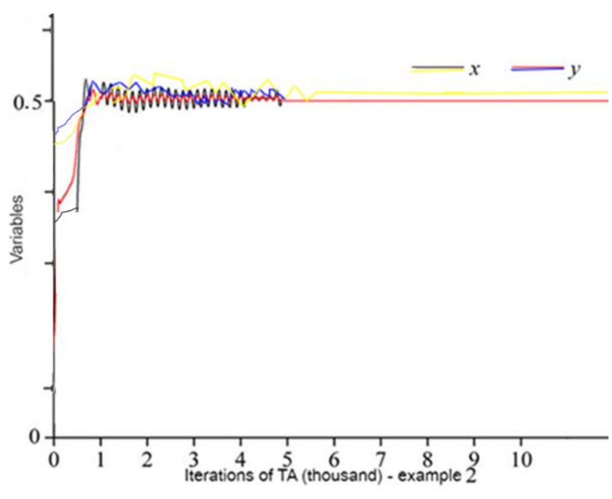

Figure 2. The transient behavior of the variables in Example 2

More problems with deferent sizes have been solved by our approach and computation results have been proposed in Table 3. According to this Table, the best solutions by our algorithm are better than the best solution by the references. The algorithm is feasible and efficient according to the Tables.

We make program with MATLAB 7.1 and use a personal computer (CPU: Intel (R) Celeron(R) 1000 M @ 1.8 GHz, RAM: 4 GB) to execute the program. References of the examples in Table 3 as follows:
Example 3 [31], Example 4 [4], Example 5 [32], Example 6 [33]. Example 3 is minimization and examples 4, 5, 6 are maximization problems.

Table 3 comparison optimal solutions and elapsed time with deferent Examples 3-6 of BLPP

	Best solution by our method ε=0.001	***Best solution according to reference [4,31-33]***	***Optimal solution***
Example 3	(1.889,0.888,0)	(1.883,0.891,0.003)	$(17/9, 8/9, 0)$
Example 4	(0,0)	(0,0)	(0,0)
Example 5	(1,0)	(1,0)	(1,0)
Example 6	(0,0.75,0,0.5,0)	(0.001,0.73,0,0.54,0)	(0,0.75,0,0.5,0)

5. Conclusion and future work

In this paper we used the KKT conditions to convert the problem into the single level problem. Then using the Fischer – Burmeister function the problem is made simpler and convert to smooth programming problem. Finally using proposed algorithm based on Taylor theorem the smoothed problem was solved. Comparing with the results of previous methods, our algorithm has better numerical results and presents better solution in less time. Also the best solution produced by proposed algorithm is feasible unlike the previous best solution by references. In the future works, the following should be researched:

(1) Examples in larger sizes can be supplied to illustrate the efficiency of the proposed algorithm.
(2) Show the efficiency of the proposed algorithm for solving other kind of the BLP such as quadratic and non-linear BLP.

References

[1] Bard, J. F. "Some properties of the bilevel programming problem." Journal of optimization theory and applications 68.2 (1991): 371-378.
[2] Vicente, Luis, Gilles Savard, and Joaquim Júdice. "Descent approaches for quadratic bilevel programming." Journal of Optimization Theory and Applications 81.2 (1994): 379-399.

[3] Lv, Yibing, et al. "A penalty function method based on Kuhn–Tucker condition for solving linear bilevel programming." Applied Mathematics and Computation 188.1 (2007): 808-813.

[4] Allende, Gemayqzel Bouza, and Georg Still. "Solving bilevel programs with the KKT-approach." Mathematical Programming 138.1-2 (2013): 309-332.

[5] Sakawa, M., I. Nishizaki, and Y. Uemura. "Interactive fuzzy programming for multilevel linear programming problems." Computers & Mathematics with Applications 36.2 (1998): 71-86.

[6] Shih, Hsu-Shih, Young-Jou Lai, and E. Stanley Lee. "Fuzzy approach for multi-level programming problems." Computers & Operations Research 23.1 (1996): 73-91.

[7] Pramanik, Surapati, and Tapan Kumar Roy. "Fuzzy goal programming approach to multilevel programming problems." European Journal of Operational Research 176.2 (2007): 1151-1166.

[8] Sakawa, Masatoshi, and Ichiro Nishizaki. "Interactive fuzzy programming for two-level linear fractional programming problems." Fuzzy Sets and Systems 119.1 (2001): 31-40.

[9] Nocedal, Jorge, and Stephen J. Wright. Conjugate gradient methods. Springer New York, 2006.

[10] A.AL Khayyal, "Minimizing a Quasi-concave Function Over a Convex Set: A Case Solvable by Lagrangian Duality", proceedings, IEEE. International Conference on Systems, Man,and Cybemeties, Tucson AZ (1985) 661-663.

[11] Mathieu, R., L. Pittard, and G. Anandalingam. "Genetic algorithm based approach to bi-level linear programming." RAIRO. Recherche opérationnelle 28.1 (1994): 1-21.

[12] Wang, Guangmin, et al. "Global convergent algorithm for the bilevel linear fractional-linear programming based on modified convex simplex method." Systems Engineering and Electronics, Journal of 21.2 (2010): 239-243.

[13] W. T. Wend, U. P. Wen, A primal-dual interior point algorithm for solving bi-level Programming problems, Asia-Pacific J. of Operational Research, 17 (2000).

[14] Thoai, N. V., Yoshitsugu Yamamoto, and Akiko Yoshise. "Global optimization method for solving mathematical programs with linear complementarity constraints." Journal of optimization theory and applications 124.2 (2005): 467-490.

[15] Hejazi, S. Reza, et al. "Linear bilevel programming solution by genetic algorithm." Computers & Operations Research 29.13 (2002): 1913-1925.

[16] Wang, Guangmin, et al. "Genetic algorithm based on simplex method for solving linear-quadratic bilevel programming problem." Computers & Mathematics with Applications 56.10 (2008): 2550-2555.

[17] Hu, Tiesong, et al. "A neural network approach for solving linear bilevel programming problem." Knowledge-Based Systems 23.3 (2010): 239-242.

[18] B. Baran Pal, D .Chakraborti , P. Biswas, "A Genetic Algorithm Approach to Fuzzy Quadratic Bi-level Programming", Second International Conference on Computing, Communication and Networking Technologies. (2010).

[19] Wan, Zhongping, Guangmin Wang, and Bin Sun. "A hybrid intelligent algorithm by combining particle swarm optimization with chaos searching technique for solving nonlinear bilevel programming problems." Swarm and Evolutionary Computation 8 (2013): 26-32.

[20] Bard, Jonathan F. Practical bilevel optimization: algorithms and applications. Vol. 30. Springer, 1998.

[21] Bard, J. F. "Some properties of the bilevel programming problem." Journal of optimization theory and applications 68.2 (1991): 371-378.

[22] B. Luce, Saïd.H, Raïd.M, "One-level reformulation of the bi-level Knapsack problem using dynamic programming", Discrete Optimization 10 (2013): 1–10.

[23] S. Dempe, A.B. Zemkoho, "On the Karush–Kuhn–Tucker reformulation of the bi-level optimization problem, Nonlinear Analysis", 75 (2012): 1202–1218.

[24] Sakawa, Masatoshi, and Takeshi Matsui. "Stackelberg solutions for random fuzzy two-level linear programming through possibility-based probability model." Expert Systems with Applications 39.12 (2012): 10898-10903.

[25] Jiang, Yan, et al. "Application of particle swarm optimization based on CHKS smoothing function for solving nonlinear bilevel programming problem." Applied Mathematics and Computation 219.9 (2013): 4332-4339.

[26] Wan, Zhongping, Guangmin Wang, and Kuolin Hou. "An interactive fuzzy decision making method for a class of bilevel programming." Fuzzy Systems and Knowledge Discovery, 2008. FSKD'08. Fifth International Conference on. Vol. 1. IEEE, 2008.

[27] Lan, Kuen-Ming, et al. "A hybrid neural network approach to bilevel programming problems." Applied Mathematics Letters 20.8 (2007): 880-884.

[28] Facchinei, Francisco, Houyuan Jiang, and Liqun Qi. "A smoothing method for mathematical programs with equilibrium constraints." Mathematical programming 85.1 (1999): 107-134.

[29] Bazaraa, Mokhtar S., Hanif D. Sherali, and Chitharanjan Marakada Shetty. Nonlinear programming: theory and algorithms. John Wiley & Sons, 2013.

[30] Thomas, George Brinton, Ross L. Finney, and Maurice D. Weir. Calculus and analytic geometry. Vol. 7. Reading, MA: Addison-Wesley, 1988.

[31] Lv, Yibing, et al. "A neural network approach for solving nonlinear bilevel programming problem." Computers & Mathematics with Applications 55.12 (2008): 2823-2829.

[32] Dempe, Stephan, and Joydeep Dutta. "Is bilevel programming a special case of a mathematical program with complementarity constraints?." Mathematical programming 131.1-2 (2012): 37-48.

[33] Zhongping, Wan, Wang Guangmin, and Lv Yibing. "A dual-relax penalty function approach for solving nonlinear bilevel programming with linear lower level problem." Acta Mathematica Scientia 31.2 (2011): 652-660.

[34] Kreyszig, Erwin, Introductory functional analysis with applications, published simultaueously in Canada, 1978.

[35] Y. Zheng, J. Liu, Z. Wan, Interactive fuzzy decision making method for solving bi-level programming problem, Applied Mathematical Modelling, Volume 38, Issue 13, 1 July 2014, Pages 3136-3141.

[36] Y. Jiang, X. Li, C. Huang, X. Wu, An augmented Lagrangian multiplier method based on a CHKS smoothing function for solving nonlinear bi-level programming problems, Knowledge-Based Systems, Volume 55, January 2014, Pages 9-14.

[37] X. He, C. Li, T. Huang, C. Li, Neural network for solving convex quadratic bilevel programming problems, Neural Networks, Volume 51, March 2014, Pages 17-25.

[38] Z. Wan, L. Mao, G. Wang, Estimation of distribution algorithm for a class of nonlinear bilevel programming problems, Information Sciences, Volume 256, 20 January 2014, Pages 184-196.

[39] P. Xu, L. Wang, An exact algorithm for the bilevel mixed integer linear programming problem under three simplifying assumptions, Computers & Operations Research, Volume 41, January 2014, Pages 309-318.

[40] E. Hosseini, I.Nakhai Kamalabadi, A Genetic Approach for Solving Bi-Level Programming Problems, Advanced Modeling and Optimization, Volume 15, Number 3, 2013.

[41] E. Hosseini, I.Nakhai Kamalabadi, Solving Linear-Quadratic Bi-Level Programming and Linear-Fractional Bi-Level Programming Problems Using Genetic Based Algorithm, Applied Mathematics and Computational Intelligence, Volume 2, 2013.

Eghbal Hosseini received B.Sc. degree in Applied Mathematics from University of Razi, Kermanshah, Iran, in 2005 and M.Sc. degree in Operation Research at the Department of Science in University of Kurdistan in 2007. He is Ph.D. student in Department of Mathematics, Tehran Payame Noor University, Iran now. His research interest is on bi-level programming problems and Meta-heuristic approaches.

Isa Nakhai Kamalabadi recieved his B.Sc. degree in computer science and its applications from Shahid Beheshti University, Iran, in 1979, the M.Sc. degree in industrial engineering from Tarbiat Modares University, Iran, in 1988 and Ph.D. degree in industrial engineering from University of Toronto in 1995. He is currently working as Professor of industrial engineering at University of Kurdistan and as Joint-Professor at Tarbiat Modares University, Iran. His research interests include: Optimization, Supply chain management, Scheduling and information technology.