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Using generalised Łukasiewicz's t-norm to represent and improve fuzzy rough approximations

Iwan Tabakow

Institute of informatics, Wroclaw University of Technology
Wroclaw, Poland
iwan.tabakow@pwr.wroc.pl

Abstract

Triangular norms are a generalisation of the classical two-valued conjunction. They were originally introduced for definition of the probabilistic (statistical) metric spaces as a generalisation of the classical triangle inequality for ordinary metric spaces. The next investigations were related with axiomatic of these norms. In this paper, a new t-norm is proposed which is generalisation of the Łukaszewicz's norm. Some selected properties of this generalised t-norm are first presented. Next , it is shown a possibility of generalisation of the notions of lower and upper approximations used in fuzzy rough sets and also of obtaining better such approximations.

Keywords: Triangular Norms, Fuzzy t-Equivalence, Fuzzy Rough Sets, Approximations

1. Introduction

Triangular norms (in short: *t-norms*) are a generalisation of the classical two-valued conjunction. They were originally introduced in [5], in the framework of the probabilistic (statistical) metric spaces as a generalisation of the classical triangle inequality for ordinary metric spaces. The next investigations [11,12] were related with axiomatic of these norms. A more detailed treatment was given in [13]. For infinite valued systems the influence of fuzzy set theory [15,16] quite recently initiated the study of a whole class of such systems of many-valued logic. In fuzzy logic systems, the basic aggregation operations are performed by the logical connectives AND and OR which provide point wise implementations of the intersection and union operations. It has been well established in the literature that the appropriate characterisations of these operations in the multi-valued logic environment are the triangular norm operators [4].

The concept of the rough set, first introduced in [7] has inspired variety of research of both theoretical and practical nature. The basic idea is that conclusions are drawn with some approximation only and are not exact as in the case of classical logic. It was presented an exact mathematical formulation of the notion of approximative (rough) equality of sets in a given approximation space. In accordance with the used equivalence relations, the obtained equivalence classes either coincide or are disjoint. However, this behaviour is lost when moving on to a fuzzy t-equivalence relations [3]. Theory of fuzzy rough sets, introduced in the last work, is a very important step for studying the notions of lower and upper approximations of a given fuzzy set. This study was an extension of the previous work [10].

We observe that the introduced definitions of the above two approximations require use of some fuzzy t-equivalence relation and hence a selection of a corresponding t-norm. In applications the often used as a t-norm is the classical Łukasiewicz's such one because the notion of fuzzy t-equivalence relation is dual to that of a pseudo-metric. And so, as an appropriate was proposed the Łukaszewicz's t-norm [3]. In fact, the Łukaszewicz's t-norm is considered as one of the tree most important in fuzzy logic systems (in common with Gödel's and product logic systems) [4]. Some review of existing t-norms was given in [2].

In this paper, a new t-norm is proposed which is generalisation of the Łukaszewicz's norm. Some selected properties of this generalised t-norm are first presented. Next , it is shown a possibility of generalisation of the notions of lower and upper approximations used in fuzzy rough sets and also of obtaining better such approximations.

This paper is arranged as follows. First, some well-known notions and definitions are given. The generalised Łukasiewicz's t-norm is briefly presented in Section 3. A generalisation of the notions of lower and upper approximations used in fuzzy rough sets is described in the next Section 4.

2. Basic notations and definitions

The *t-norm operator* provides the characterisation of the AND operator. It is a binary operation $\otimes$: $[0,1]^2 \rightarrow [0,1]$ with the following properties (for any $x,y,u,v \in [0,1]$: e.g. see [2]):

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$x \otimes y = y \otimes x$	*commutative*
$x \otimes y \geq u \otimes v$ for $x \geq u$ and $y \geq v$	*monotonic*
$x \otimes (y \otimes z) = (x \otimes y) \otimes z$	*associative*
$x \otimes 1 = 1 \otimes x = x$	has 1 as *unit element*

The dual *t-conorm operator* (called also: *s-norm*), characterises the OR operator. It is a binary operation $\oplus : [0,1]^2 \rightarrow [0,1]$ having properties as follows (for any $x,y,u,v \in [0,1]$):

$x \oplus y = y \oplus x$	*commutative*
$x \oplus y \geq u \oplus v$ for $x \geq u$ and $y \geq v$	*monotonic*
$x \oplus (y \oplus z) = (x \oplus y) \oplus z$	*associative*
$x \oplus 0 = 0 \oplus x = x$	has 0 as *unit element*

In general, the notion of (continuous) *fuzzy negation* can be introduced as a function mapping $f : [0,1] \rightarrow [0,1]$ with the following properties (for any $x, y \in [0,1]$ [2]:

$f(0) = 1$ and $f(1) = 0$	*the terminal point values*
$x < y \Rightarrow f(x) \geq f(y)$	*monotonicity*
$f(f(x)) = x$	*involutivity*
$f(x)$ is a continuous function	*continuity*

It can be observed that a very simple function satisfying the above properties is the classical Łukasiewicz's negation $f_Ł(x) =_{df} 1 - x$. Some generalisations were also introduced, e.g. such as: *Sugeno's fuzzy negation* $f_S(x) =_{df} \dfrac{1 - x}{1 + \lambda x}$, where $\lambda \in (-1, \infty)$ or also *Yager's fuzzy negation* $f_Y(x) =_{df} (1 - x^{\alpha})^{1/\alpha}$ where $\alpha \in (0, \infty)$.

Let $x' =_{df} f(x)$ be a continuous fuzzy negation. So any t-conorm is dual to the corresponding t-norm under the order-reversing operation which assigns x' to x on $[0,1]$. And hence, for a given t-norm the complementary conorm is defined as follows (a generalisation of *De Morgan's laws*): $x \oplus y =_{df} (x' \otimes y')'$. The Yager's fuzzy negation is assumed below.

Equivalence relations and orderings are key concepts of mathematics and they play a fundamental role in the areas of fuzzy logic and fuzzy systems, e.g. for interpretation of fuzzy partitions and fuzzy controllers [1] or also for construction of lower and upper approximations of fuzzy sets [3] and so on.

Consider a given set X. Let $\rho : X \times X \rightarrow [0,1]$ be a *fuzzy relation defined in* X and $\otimes$ be a given t-norm. We shall say ρ is a *fuzzy t-equivalence* iff it is at the same time *reflexive*, *symmetric* and *t-transitive*, i.e. $\rho(x,x) = 1$, $\rho(x,y) = \rho(y,x)$, and $\rho(x,z) \geq \rho(x,y) \otimes \rho(y,z)$, respectively (for any $x, y, z \in X$) [1].

Example 1

The following example fuzzy t-equivalence relation was considered in [3]: $\rho(x,y) =_{df} \max\{0, 1 - |x - y|\}$. Equivalently we can obtain: $\rho(x,y) = 1 - \min\{1, |x - y|\}$. Obviously, the above relation is reflexive and symmetric. And it is t-transitive under the classical Łukasiewicz's t-norm: $x \otimes y =_{df} \max\{0, x + y - 1\}$, where $X =_{df} \mathbb{R}$ (the set of real numbers). The proof of the t-transitivity property is omitted (a more generalised proof is given in the next Section).

Consider a finite subset of integers $Y \subsetneq \mathbb{R}$. Let ρ be defined in Y. We have: $\rho(x,y) =$ if $x = y$ then 1 else 0. The obtained membership matrix M_ρ is an identity matrix, i.e. a square matrix with ones on the main diagonal and zeros elsewhere. This problem can be omitted by using some normalisation, e.g. $\rho_\sigma(x,y) =_{df} 1 - \min\{1, \sigma|x - y|\}$, where σ is the reciprocal of the largest value of $|x - y|$, i.e. $\sigma =_{df} 1 / \max\{|x - y| / x, y \in Y; x \neq y\}$ (the Chebyshev's distance with $x \neq y$).

For example, let us consider the set $Y =_{df} \{x_1, x_2, x_3, x_4\}$, where $x_i =_{df} i$ $(i = 1,2,3,4)$. Since $\sigma = 1/3$, the following membership matrix associated with the fuzzy t-equivalence relation ρ_σ can be obtained: $M_{\rho_\sigma} = \begin{bmatrix} 1 & 2/3 & 1/3 & 0 \\ 2/3 & 1 & 2/3 & 1/3 \\ 1/3 & 2/3 & 1 & 2/3 \\ 0 & 1/3 & 2/3 & 1 \end{bmatrix}$. □

Any reflexive and symmetric fuzzy relation is said to be a *fuzzy similarity relation*. In particular, fuzzy similarity relations may be associated with some distance functions, e.g. of the Minkowski class, Canberra, squared chord, squared Chi-square, cosine and so on. Moreover, a linear convex combination of a finite number of fuzzy similarity relations is also a fuzzy similarity relation. And hence, to obtain a fuzzy t-equivalence relation usually a t-transitivity closure algorithm should be realised. All these considerations are omitted here.

3. The generalised Łukasiewicz's t-norm

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Consider the following two *similar* Abelian systems, i.e. of the same type (0,0,2): $\mathscr{A}_1 =_{df} ([0,1] ; 1, 0 ; \hat{\otimes})$ and $\mathscr{A}_2 =_{df} ([0,1] ; 1, 0 ; \otimes)$, where $\hat{\otimes}$ and $\otimes$ are two t-norms. We shall assume that $\otimes$ is a priori given t-norm called *source t-norm* (or *prototypical representative*). Assume that $f : [0,1] \to [0,1]$ is a given increasing bijection and $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic with respect to f. Hence, the following two conditions should be satisfied (for any $x, y \in [0,1]$):

(i) $f(1) = 1$, $f(0) = 0$ (the *algebraic constants preservation*)

(ii) $f(x \hat{\otimes} y) = f(x) \otimes f(y)$ (the *algebraic operations preservation*).

Since f is bijection and in accordance with the above assumptions, there exists an inverse function f^{-1} (having the same properties as the original function f) such that: $f^{-1} (f(x \hat{\otimes} y)) = f^{-1} (f(x) \otimes f(y))$. Therefore, the new $\hat{\otimes}$ can be obtained in an unique way by the following well-known equality: $x \hat{\otimes} y =_{df} f^{-1}(f(x) \otimes f(y))$.

Let now consider the increasing bijection $y = f(x) =_{df} x^{\alpha}$ defined in [0,1]. The inverse function $y = f^{-1}(x) =_{df} \sqrt[\alpha]{x}$, where $x \geq 0$ and $\alpha > 0$. It is selected as a source t-norm the Łukasiewicz's one, i.e. $x \otimes y =_{df} \max\{0, x + y - 1\}$. And hence, the following t-norm can be obtained: $x \hat{\otimes} y =_{df} (\max\{0, x^{\alpha} + y^{\alpha} - 1\})^{1/\alpha} = \max\{0, x^{\alpha} + y^{\alpha} - 1\}^{1/\alpha}$.

The following properties are satisfied (because of space limitations some proofs are omitted here) [14].

Proposition 1

The above systems $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic with respect to $y = x^{\alpha}$ defined in [0,1]. ▫

Proposition 2

Let $x' =_{df} (1 - x^{\alpha})^{1/\alpha}$ be the Yager's fuzzy negation. Then, the following t-conorm can be obtained: $x \hat{\oplus} y =_{df} \min\{1, x^{\alpha} + y^{\alpha}\}^{1/\alpha}$. ▫

It is easily to show that the above two generalised norms are well-defined and the corresponding axioms are satisfied (this is omitted). In particular, since $\hat{\otimes}$ and $\hat{\oplus}$ are associative they can be generalised for more than two (a finite number) arguments. In fact, the following proposition is satisfied.

Proposition 3

The generalised n-argument t-norm and t-conorm are presented as follows:

$\hat{\otimes}_{i=1}^{n} x_i = \max\{0, \sum_{i=1}^{n} x_i^{\alpha} - n + 1\}^{1/\alpha}$ and

$\hat{\oplus}_{i=1}^{n} x_i = \min\{1, \sum_{i=1}^{n} x_i^{\alpha}\}^{1/\alpha}$, respectively. ▫

The *fuzzy implication* connective is sometimes disregarded but is of fundamental importance for fuzzy logic in the narrow sense. A straightforward but logically less interesting possibility is to define implication from disjunction and negation or conjunction and negation using the corresponding theses of classical logic. Such implications are called *S-implications.* In fact, more useful and interesting are the so-called R-implications and any such implication can be interpreted as a binary operation over [0,1] and specified as a residuum of the corresponding t-norm. It was shown that this residuum is unique if the considered t-norm is at least left-continuous. In general, the logical value of any *R-implication* can be defined as follows [4]: $x \Rightarrow y =_{df} \sup\{z \in [0,1] / x \otimes z \leq y\}$ (for any $x, y \in [0,1]$ and any left-continuous $\otimes$).

Proposition 4

Let $\hat{\otimes}$ be the above introduced t-norm. The fuzzy implication $x \Rightarrow_{\alpha} y$ having logical value as follows: $x \Rightarrow_{\alpha} y =_{df}$ if $x \leq y$ then 1 else $(1 - x^{\alpha} + y^{\alpha})^{1/\alpha}$ is a well-defined and unique R-implication, where $\alpha > 0$. ▫

Since x^{α} is increasing in [0,1] then $x^{\alpha} \leq y^{\alpha}$ if $x \leq y$. And hence, $y^{\alpha} - x^{\alpha} \geq 0$. Then $1 + y^{\alpha} - x^{\alpha} \geq 1$ and $(1 - x^{\alpha} + y^{\alpha})^{1/\alpha} \geq 1$. In a similar way, assuming $x > y$ we can obtain $(1 - x^{\alpha} + y^{\alpha})^{1/\alpha} < 1$.

And so, the following corollary is satisfied.

Corollary 1

$x \Rightarrow_{\alpha} y = \min\{1, 1 - x^{\alpha} + y^{\alpha}\}^{1/\alpha}$. ▫

An additional advantage of the classical Łukasiewicz's t-norm is the coincidence of the corresponding S- and R-implications [3]. And this property is also satisfied in the case of using a generalised Łukasiewicz's t-norm.

Corollary 2

The S- and R-implications coincide if Yager's fuzzy negation is assumed.

Proof:

$x \Rightarrow_{\alpha} y =_{df} (x \hat{\otimes} y')'$

$$= (1 - (\max\{0, x^{\alpha} + ((1 - y^{\alpha})^{1/\alpha})^{\alpha} - 1\}^{1/\alpha})^{\alpha})^{1/\alpha}$$
$$= (1 - \max\{0, x^{\alpha} - y^{\alpha}\})^{1/\alpha}$$
$$= \min\{1, 1 - x^{\alpha} + y^{\alpha}\}^{1/\alpha} . \square$$

The relations between the Zadeh's t-norm and t-conorm and the presented ones are given in the next proposition ('iff ' denotes 'if and only if').

Proposition 5

$\min\{x,y\} \geq \max\{0, x^{\alpha} + y^{\alpha} - 1\}^{1/\alpha}$ and $\max\{x,y\} \leq \min\{1, x^{\alpha} + y^{\alpha}\}^{1/\alpha}$

Proof:

Assume that $x \leq y$. Then $\min\{x,y\} = x$ and $\max\{x,y\} = y$ ($x, y \in [0,1]$). Since $x \geq 0$ and $y \leq 1$ it is sufficient to show that $x \geq (x^{\alpha} + y^{\alpha} - 1)^{1/\alpha}$ and $y \leq (x^{\alpha} + y^{\alpha})^{1/\alpha}$. We have: $x \geq (x^{\alpha} + y^{\alpha} - 1)^{1/\alpha}$ iff $x^{\alpha} \geq x^{\alpha} + y^{\alpha} - 1$ iff $y^{\alpha} \leq 1$. On the other hand, $y \leq (x^{\alpha} + y^{\alpha})^{1/\alpha}$ iff $y^{\alpha} \leq x^{\alpha} + y^{\alpha}$ iff $x^{\alpha} \geq 0$ (the proof for $x > y$ is omitted). $\square$

Therefore, $\max\{0, x^{\alpha} + y^{\alpha} - 1\}^{1/\alpha} \leq \min\{1, x^{\alpha} + y^{\alpha}\}^{1/\alpha}$. It can be shown the binary operation $\hat{\otimes}$ is a nilpotent Archimedean t-norm. Obviously, the classical Łukasiewicz's system (t-norm, t-conorm, fuzzy negation and implication) can be obtained assuming $\alpha = 1$. It can be observed that the graph of $\hat{\otimes}$ (i.e. of the two-argument function $z = x \hat{\otimes} y$) and this one associated with Łukasiewicz's t-norm are different. In fact, assuming $z = 0$, all points of plane XOY corresponding to the Yager's negation $y = (1 - x^{\alpha})^{1/\alpha}$ will be located on the left side and the right side of the line $y = 1 - x$ (the Łukasiewicz's negation), depending on the used values for α ($\alpha < 1$ or $\alpha > 1$, respectively). And the last two functions will coincide with $\alpha = 1$. In accordance with Proposition 1, $\hat{\otimes}$ is a continuous t-norm (it is a superposition of continuous functions and the sup-preservation property is satisfied for $\hat{\otimes}$). Moreover, there is no any idempotent $x \in (0,1)$ and hence $\hat{\otimes}$ is Archimedean t-norm (since $x \hat{\otimes} x \neq x$, i.e. $\max\{0,2x^{\alpha} - 1\} \neq x^{\alpha}$, for any $x \in (0,1)$).

As an illustration, the above introduced generalised Łukasiewicz's t-conorm can be used for obtaining a distance function of the Minkowski class (e.g. see [6]).

Example 2

Let now $X =_{df} \{\mathbf{x}_1, \mathbf{x}_2, \ldots ,\mathbf{x}_n\} \subseteq \mathbb{R}^p$ be a given finite set of p-component vectors and $\mathbf{x}_i, \mathbf{x}_k \in X$. Consider the following expression: $1 - \hat{\oplus}_{j=1}^{p} |x_{ij} - x_{kj}|$. Hence, in accordance with Proposition 3 we can obtain: $1 - \min\{1, \sigma \cdot \sum_{j=1}^{p} |x_{ij} - x_{kj}|^{\alpha}\}^{1/\alpha}$, where σ is the reciprocal of the largest value of the sum $\sum_{j=1}^{p} |x_{ij} - x_{kj}|^{\alpha}$, i.e. $\sigma =_{df} 1 / \max\{ \sum_{j=1}^{p} |x_{ij} - x_{kj}|^{\alpha} / \mathbf{x}_i, \mathbf{x}_k \in X; \mathbf{x}_i \neq \mathbf{x}_k\}$. And finally, the following distance function can be obtained: $\hat{\rho}(\mathbf{x}_i,\mathbf{x}_k) =_{df} 1 - \min\{\sigma 1 \sum_{j=1}^{p} |x_{ij} - x_{kj}|^{\alpha}\}^{1/\alpha}$. It is easily to show that $\hat{\rho}(\mathbf{x}_i,\mathbf{x}_k)$ equals to the distance function of the Minkowski class. $\square$

We observe the one-dimensional fuzzy t-equivalence ρ_{σ} , introduced in the previous Example 1, can be interpreted as a particular case of distance function of the Minkowski class. Moreover, the considered in [3] relation $\rho(x,y) =_{df} \max\{0, 1 - |x - y|\}$ remains a fuzzy t-equivalence for any $\alpha \geq 1$. In fact, the following proposition holds.

Proposition 6

Let $\alpha \geq 1$. Then $\rho(\mathbf{x},\mathbf{y}) =_{df} \max\{0, 1 - |x - y|\}$ is a fuzzy t-equivalence with respect to the generalised Łukasiewicz's t-norm.

Proof:

Assume that $\alpha \geq 1$. It is sufficient to show that ρ is t-transitive, i.e. $\rho(x,z) \geq \rho(x,y) \hat{\otimes} \rho(y,z)$ (for any $x, y, z \in \mathbb{R}$), where $\hat{\otimes}$ is the generalised Łukasiewicz's t-norm. Equivalently, the following inequality should be shown: $\max\{0, 1 - |x - z|\} \geq \max\{0, \max\{0, 1 - |x - y|\}^{\alpha} + \max\{0, 1 - |y - z|\}^{\alpha} - 1\}^{1/\alpha}$. And hence: $\max\{0, 1 - |x - z|\}^{\alpha} \geq \max\{0, \max\{0, 1 - |x - y|\}^{\alpha} + \max\{0, 1 - |y - z|\}^{\alpha} - 1\}$. Since any absolute value $|x - z|, |x - y|$ and $|y - z|$ may be greater than, equal to, or less than 1, in general, $3^3 = 27$ cases should be considered (eventually reduced to $2^3 = 8$). However, the most important is the case when $|x - z| = 1$, $|x - y| < 1$ and $|y - z| < 1$. Hence, the following inequality should be shown: $(1 - |x - y|)^{\alpha} + (1 - |y - z|)^{\alpha} \leq 1$. This case is considered below.

Since $1 = |x - z| \leq |x - y| + |y - z|$, $|x - y|, |y - z| < 1$ and $1 - |x - z| \geq 1 - (|x - y| + |y - z|)$, the above inequality is always satisfied. In fact, for any $\alpha \geq 1$ we have: $(1 - |x - y|)^{\alpha} + (1 - |y - z|)^{\alpha} \leq (1 - |x - y| + 1 - |y - z|)^{\alpha} = (1 - (|x - y| + |y -$

$z|) + 1)^{\alpha} \leq (1 - |x - z| + 1)^{\alpha} = (1 - 1 + 1)^{\alpha} = 1^{\alpha} = 1$ (since $\lfloor\alpha\rfloor \leq \alpha \leq \lceil\alpha\rceil$, where $\lfloor\alpha\rfloor$ and $\lceil\alpha\rceil$ are the corresponding floor and ceiling functions and $a^n + b^n \leq (a + b)^n$). ▫

4. The generalised fuzzy rough approximations

The concept of the rough set, first introduced in [7] has inspired variety of research of both theoretical and practical nature. The basic idea is that conclusions are drawn with some approximation only and are not exact as in the case of classical logic. It was presented an exact mathematical formulation of the notion of approximative (rough) equality of sets in a given *approximation space*, understood as a pair $A =_{df} (U, \rho)$, where U is a certain set called *universe* and $\rho \subseteq U \times U$ is an equivalence relation. The rough set concept can be of some importance, primarily in some branches of artificial intelligence, such as inductive reasoning, automatic classification, pattern recognition, information systems and decision tables, state identification, learning algorithms, cluster analysis, measurement theory, taxonomy, and so on [8,9].

In general, the lower and upper approximations of a given subset $X \subseteq U$ are computed using ρ and defined as follows: $\underline{A}(X) =_{df} \{x \in U / [x]_\rho \subseteq X\}$ and $\overline{A}(X) =_{df} \{x \in U / [x]_\rho \cap X \neq \varnothing\}$, respectively. Equivalently, we have: $\underline{A}(X) =_{df} \bigcup\{[x]_\rho \in U/\rho \ / \ [x]_\rho \subseteq X\}$ and $\overline{A}(X) =_{df} \bigcup\{[x]_\rho \in U/\rho \ / \ [x]_\rho \cap X \neq \varnothing\}$. Obviously, the obtained *equivalence classes* (called also *elementary sets* or *atoms*) $[x]_\rho$ in U/ρ (the *quotient set*) either coincide or are disjoint.

An equivalent version of the above two approximations was originally proposed in [3]. And so, we have: $y \in \underline{A}(X) \Leftrightarrow_{df} \forall_{x \in U}(x \rho y \Rightarrow x \in X)$ and $y \in \overline{A}(X) \Leftrightarrow_{df} \exists_{x \in U}(x \rho y \wedge x \in X)$.

By definition, it follows that $y \in \underline{A}(X) \Leftrightarrow [y]_\rho \subseteq X$ and $y \in \overline{A}(X) \Leftrightarrow [y]_\rho \cap X \neq \varnothing$. And hence, the following two properties should be satisfied: $[y]_\rho \subseteq X \Leftrightarrow \forall_{x \in U}(x \rho y \Rightarrow x \in X)$ and $[y]_\rho \cap X \neq \varnothing \Leftrightarrow \exists_{x \in U}(x \rho y \wedge x \in X)$. A more formal treatment is given below.

The following designations are used in the next proofs (the names associated with some primitive and/or derived rules are in accordance with the corresponding Łukasiewicz´s symbols): '– A' (*rule of omitting a disjunction*), '– K' (*rule of omitting a conjunction*), '– C' (*rule of detachment for implication* or *omitting an implication*), 'NC' (*rule of negating an implication*), 'NK' (*rule of negating a conjunction*), 'SR'(*rule of substitution for equivalence*), '$-\forall^*$', '$-\exists^*$', '$N\forall^*$' and '$N\exists^*$' (*rules of omitting an universal* and *an existential bounded quantifiers and also negating an universal* and *an existential bounded quantifiers, respectively*). The introduced abbreviations 'a', 'aip', and 'contr.', denote: *assumption*(*s*), *assumption*(*s*) *of indirect proof*, and *contradiction*, respectively. Provided there is no ambiguity and depending on the context, by 'a' it is also denoted an element of U. Obviously, any element belongs to U and any set of such elements is subset of U.

Proposition 7

$$[y]_\rho \subseteq X \Leftrightarrow \forall_{x \in U}(x \rho y \Rightarrow x \in X)$$

Proof (*if-condition*):

(1)	$[y]_\rho \subseteq X$	{a}
(2)	$\sim \forall_{x \in U}(x \rho y \Rightarrow x \in X)$	{aip}
(3)	$\exists_{x \in U}(x \rho y \wedge x \notin X)$	{ $N\forall^*$, NC, SR: 2}
(4)	$a \in U$	
(5)	$a \rho y$	{$-\exists^*$, – K : 3}
(6)	$a \notin X$	
(7)	$a \in [y]_\rho$	{df.' $[y]_\rho$': 5}
(8)	$a \in X$	{df.' $\subseteq$': 1,7}
	contr. ▫	{6,8}

Proof (only *if-condition*):

(1)	$\forall_{x \in U}(x \rho y \Rightarrow x \in X)$	{a}
(2)	$[y]_\rho \not\subseteq X$	{aip}
(3)	$\sim([y]_\rho \subseteq X)$	{df.' $\not\subseteq$': 2}
(4)	$\sim \forall_{x \in U}(x \in [y] \Rightarrow x \in X)$	{df.' $\subseteq$', SR : 3}
(5)	$\exists_{x \in U}(x \in [y]_\rho \wedge x \notin X)$	{ $N\forall^*$, NC, SR : 4}
(6)	$a \in U$	
(7)	$a \in [y]_\rho$	{$-\exists^*$, – K : 5}
(8)	$a \notin X$	
(9)	$a \in U \Rightarrow (a \rho y \Rightarrow a \in X)$	{$-\forall^*$: 1}
(10)	$a \rho y \Rightarrow a \in X$	{– C : 6,9}
(11)	$a \rho y$	{df.' $[y]_\rho$': 7}
(12)	$a \in X$	{– C : 10,11}
	contr. ▫	{8,12}

Proposition 8

$$[y]_\rho \cap X \neq \emptyset \Leftrightarrow \underset{x \in U}{\exists}(x \rho y \wedge x \in X)$$

Proof (*if-condition*):

(1)	$[y]_\rho \cap X \neq \emptyset$	{a}
(2)	$\sim \underset{x \in U}{\exists}(x \rho y \wedge x \in X)$	{aip}
(3)	$\underset{x \in U}{\forall}(x \rho' y \vee x \notin X)$	{ N$\exists^*$, NK, SR : 2}
(4)	$\underset{x \in U}{\exists}(x \in [y]_\rho \cap X)$	{1}
(5)	$a \in U$	
(6)	$a \in [y]_\rho$	{$-\exists^*$, df.'$\cap$', – K : 4}
(7)	$a \in X$	
(8)	$a \rho y$	{df.' $[y]_\rho$': 6}
(9)	$a \in U \Rightarrow a \rho' y \vee a \notin X$	{$-\forall^*$: 3}
(10)	$a \rho' y \vee a \notin X$	{– C : 5,9}
(11)	$a \notin X$	{– A : 8,10}
	contr. ▫	{7,11}

Proof (only *if-condition*):

(1)	$\underset{x \in U}{\exists}(x \rho y \wedge x \in X)$	{a}
(2)	$[y]_\rho \cap X = \emptyset$	{aip}
(3)	$\underset{x \in U}{\forall}(x \notin [y]_\rho \cap X)$	{2}
(4)	$a \in U$	{$-\exists^*$, – K : 1}
(5)	$a \rho y$	
(6)	$a \in X$	
(7)	$a \notin [y]_\rho \cap X$	{$-\forall^*$: 3}
(8)	$a \notin [y]_\rho \vee a \notin X$	{ df.'$\cap$', NK, SR : 7}
(9)	$a \notin [y]_\rho$	{– A : 6,8}
(10)	$a \rho' y$	{df.' $[y]_\rho$': 9}
	contr. ▫	{5,10}

Let $X \subseteq U$. We shall say that X is *exact* (or *measurable*) in A if and only if $\underline{A}(X) = \bar{A}(X)$. And hence, X is exact in A if and only if X is a *composed set* in A, i.e. a finite union of elementary sets. Any $X, Y \subseteq U$ are said to be *roughly equal* (*roughly bottom-equal* or *roughly top-equal*) in A if and only if X and Y have the same lower and upper approximations (either the same lower approximations or the same upper approximations) in A. It is easily to show the above notions of 'roughly equal', 'roughly bottom-equal' and 'roughly top-equal' are equivalence relations on P(U) (the *powerset* of U). The corresponding equivalence classes are said to be *rough* (*lower*, *upper*) *sets*. Therefore, if X is not exact in A, then X will belong to some subfamily $\subseteq$ P(U) called *rough set* [8].

In accordance to the above considerations, any rough set is related to some ordered pair (X_1, X_2), where $X_1 =_{df} \underline{A}(X)$ and $X_2 =_{df} \bar{A}(X)$ [3,10]. The proposed here predicate-oriented version for $\underline{A}(X)$ and $\bar{A}(X)$ was extended in the area of fuzzy sets. More exactly, the lower and upper approximations associated with any fuzzy set can be constructed by means of the notions of a fuzzy implication, a t-norm and a fuzzy t-equivalence. And hence, the following *lower* and *upper approximations of a fuzzy set* μ in U were introduced [10]. Provided there is no ambiguity and for convenience, here the domain of a fuzzy set X is denoted by U.

$$\underline{\mu}(y) =_{df} \inf\{x \in U / \rho(x,y) \Rightarrow \mu(x)\}$$

$$\bar{\mu}(y) =_{df} \sup\{x \in U / \rho(x,y) \otimes \mu(x)\}$$

Let now ρ be a fuzzy t-equivalence in U and $y \in U$. The ρ - *foreset* of y is the fuzzy set ρy having membership function $\rho y(x) =_{df} \rho(x,y)$ for all $x \in U$.

Example 3

Consider the fuzzy t-equivalence relation ρ_σ of Example 1. In accordance with Proposition 6, this property of ρ_σ is preserved assuming $\alpha \geq 1$. As an example, the ρ_σ - foresets of x_1 and x_4 are disjoint, i.e. they have an empty t-intersection. In fact, we have: $(\rho_\sigma x_1 \cap \rho_\sigma x_4)(x_i) = \rho_\sigma x_1(x_i) \hat{\otimes} \rho_\sigma x_4(x_i) = \rho_\sigma(x_i,x_1) \hat{\otimes} \rho_\sigma(x_i,x_4) = \max\{0, \rho_\sigma(x_i,x_1)^\alpha + \rho_\sigma(x_i,x_4)^\alpha - 1\}^{1/\alpha} = 0$ (for any $\alpha \geq 1$ and $i = 1,2,3,4$), e.g. $(\rho_\sigma x_1 \cap \rho_\sigma x_4)(x_3) = \max\{0, (1/3)^\alpha + (2/3)^\alpha - 1\}^{1/\alpha} = 0$ (since $2^\alpha \leq 3^\alpha - 1$, for $\alpha \geq 1$). The corresponding rows for x_1 and x_4 in M_{ρ_σ} have all elements different.

Let now consider the ρ_σ - foresets of x_1 and x_3. Since $\rho_\sigma x_1(x_2) = \rho_\sigma x_3(x_2) = 2/3$ then $(\rho_\sigma x_1 \cap \rho_\sigma x_3)(x_2) = \max\{0, (2/3)^\alpha + (2/3)^\alpha - 1\}^{1/\alpha}$, e.g. the value of this maximum is equal to 1/3 (for $\alpha = 1$) or 0 ($\alpha = 2$). Hence, in the first case, $x_2 = 2$ belongs to degree 1/3 to the t-intersection of the above two ρ_σ - foresets, i.e. they are not disjoint. ▫

Let ρ be a fuzzy relation that models an approximate equality. Then, we shall say that ρy is a *fuzzy similarity class* of y. According to the last example, an element $y \in \rho y$ can also belong to other, different similarity classes to a certain degree. In fact, the following list of candidate definitions for the lower (the upper) approximation of μ should be considered [3].

Any $y \in U$ belongs to the lower (the upper) approximation of μ to the degree to which:

a. All fuzzy similarity classes containing y are included in μ (have a nonempty intersection with μ),

b. At least one fuzzy similarity class containing y is included in μ (has a nonempty intersection with μ) and

c. The fuzzy similarity class ρy is included in μ (has a nonempty intersection with μ).

In accordance with the above considerations, the notions of tight, loose and usual lower and upper approximations were introduced. For convenience, the following designations for tight, loose and usual lower (upper) approximations are used below: $\underline{\mu}_t$, $\underline{\mu}_l$ and $\underline{\mu}_u$ ($\bar{\mu}_t$, $\bar{\mu}_l$ and $\bar{\mu}_u$), respectively. In the case of usual approximations the index 'u' may be omitted. The following extended versions can be proposed (for all $y \in U$).

The *tight*, *loose* and *usual lower approximations*:

$\underline{\mu}_t(y) =_{df} \inf\{z \in U / \rho z(y) \Rightarrow_\alpha \inf\{x \in U / \rho z(x) \Rightarrow_\alpha \mu(x)\}\}$,

$\underline{\mu}_l(y) =_{df} \sup\{z \in U / \rho z(y) \hat{\otimes} \inf\{x \in U / \rho z(x) \Rightarrow_\alpha \mu(x)\}\}$,

$\underline{\mu}_u(y) =_{df} \inf\{x \in U / \rho y(x) \Rightarrow_\alpha \mu(x)\}$.

The *tight*, *loose* and *usual upper approximations*:

$\bar{\mu}_t(y) =_{df} \inf\{z \in U / \rho z(y) \Rightarrow_\alpha \sup\{x \in U / \rho z(x) \hat{\otimes} \mu(x)\}\}$,

$\bar{\mu}_l(y) =_{df} \sup\{z \in U / \rho z(y) \hat{\otimes} \sup\{x \in U / \rho z(x) \hat{\otimes} \mu(x)\}\}$,

$\bar{\mu}_u(y) =_{df} \sup\{ x \in U / \rho y(x) \hat{\otimes} \mu(x)\}$.

According to the last definitions, $\hat{\otimes}$ and $\Rightarrow_\alpha$ denote the generalised Łukasiewicz's t-norm and fuzzy implication (Proposition 4, Corollaries 1 and 2), respectively.

Example 4

Let U be the subset Y from Example 1 and ρ_σ be the obtained fuzzy t-equivalence represented by M_{ρ_σ} . Assume that $\mu =_{df} (3/5, 4/5, 1/5, 2/5)$ is a fuzzy set defined in Y. In accordance with the above definitions, e.g. the following usual lower and usual upper approximations are obtained (the index 'u' is omitted here).

α	$\underline{\mu}$	$\bar{\mu}$
1	(3/5, 8/15, 1/5, 2/5)	(3/5, 4/5, 7/15, 2/5)
2	(3/5, $\sqrt{134}$ /15, 1/5,2/5)	(3/5, 4/5, $\sqrt{19}$ /15, 2/5)

As an illustration, the computations related to $\underline{\mu}(x_3)$ and $\bar{\mu}(x_3)$, i.e. $y =_{df} x_3$ and $\alpha =_{df} 2$, are given below. And so, in the case of the usual lower approximation we can obtain.

$\rho(x_1,x_3) \Rightarrow_\alpha \mu(x_1) = 1/3 \Rightarrow_\alpha 3/5 = \min\{1, 1 - (1/3)^2 + (3/5)^2\}^{1/2} = 1$,

$\rho(x_2,x_3) \Rightarrow_\alpha \mu(x_2) = 2/3 \Rightarrow_\alpha 4/5 = 1$,

$\rho(x_3,x_3) \Rightarrow_\alpha \mu(x_3) = 1 \Rightarrow_\alpha 1/5 = 1/5$,

$\rho(x_4,x_3) \Rightarrow_\alpha \mu(x_4) = 2/3 \Rightarrow_\alpha 2/5 = \sqrt{161}/15$.

And hence: $\underline{\mu}(x_3) = \min\{1, 1, 1/5, \sqrt{161}/15\} = 1/5$ (since $3^2 < 161$).

In a similar way, in the case of the usual upper approximation we have.

$\rho(x_1,x_3) \hat{\otimes} \mu(x_1) = 1/3 \hat{\otimes} 3/5 = \max\{0, (1/3)^2 + (3/5)^2 - 1\}^{1/2} = 0$,

$\rho(x_2,x_3) \hat{\otimes} \mu(x_2) = 2/3 \hat{\otimes} 4/5 = \sqrt{19}/15$,

$\rho(x_3,x_3) \hat{\otimes} \mu(x_3) = 1 \hat{\otimes} 1/5 = 1/5$,

$\rho(x_4,x_3) \hat{\otimes} \mu(x_4) = 2/3 \hat{\otimes} 2/5 = 0$.

Therefore: $\bar{\mu}(x_3) = \max\{0, \sqrt{19}/15, 1/5, 0\} = \sqrt{19}/15$ (since $3^2 < 19$). We observe a better approximation using $\alpha = 2$, i.e. the obtained Hamming distance: $d(\underline{\mu}, \bar{\mu}) =_{df} \sum_{x \in Y} |\underline{\mu}(x) - \bar{\mu}(x)|$ is less than this one for $\alpha = 1$ (the classical case). And so: $\underline{\mu}/_{\alpha = 1} \subseteq \underline{\mu}/_{\alpha = 2} \subseteq \mu \subseteq \bar{\mu}/_{\alpha = 2} \subseteq \bar{\mu}/_{\alpha = 1}$. □

Proposition 9

$\underline{\mu}/_{\alpha = 1} \subseteq \underline{\mu}/_{\alpha = 2} \subseteq \mu \subseteq \bar{\mu}/_{\alpha = 2} \subseteq \bar{\mu}/_{\alpha = 1}$

Proof:

It is sufficient to show that: (a) $\min\{1, 1 - x + y\}^2 \leq \min\{1, 1 - x^2 + y^2\}$ and (b) $\max\{0, x + y - 1\}^2 \geq \max\{0, x^2 + y^2 - 1\}$.

a) Let $x \leq y$. Hence: $x^2 \leq y^2$, $1 - x + y \geq 1$, $1 - x^2 + y^2 \geq 1$ and the left (L) and right (R) sides coincide, $L = R = 1$.

Assume now that $x > y$. We have: $L = (1 - x + y)^2$ and $R = 1 - x^2 + y^2$. It is necessary to show that: $(1 - x + y)^2 \leq 1 - x^2 + y^2$ $(x, y \in [0,1], x > y)$. And so, we have: $(1 - x + y)^2 \leq 1 - x^2 + y^2$ iff $(1 + (y - x))^2 \leq 1 + y^2 - x^2$ iff $1 + 2y - 2x + y^2 - 2yx + x^2 \leq 1 + y^2 - x^2$ iff $2y - 2x - 2yx + 2x^2 \leq 0$ iff $y - x - x(y - x) \leq 0$ iff $(y - x)(1 - x) \leq 0$. Since $x > y$ and $x \leq 1$ then $y - x < 0$ and $1 - x \geq 0$. Hence, the last inequality is always satisfied.

b) Let $x + y \leq 1$. Since $x + y - 1 \leq 0$ the left side $L = 0$. Also $(x + y)^2 \leq 1$ and hence $x^2 + y^2 \leq 1$ $(2xy \geq 0$, for $x, y \geq 0)$. Then: $R = 0$.

Let now $x + y > 1$. Since $x + y - 1 > 0$ then $L = (x + y - 1)^2 > 0$. And hence, it is sufficient to show that $(x + y - 1)^2 \geq x^2 + y^2 - 1$. We have: $(x + y - 1)^2 \geq x^2 + y^2 - 1$ iff $((x + y) - 1)^2 \geq x^2 + y^2 - 1$ iff $x^2 + y^2 + 2xy - 2x - 2y + 1 \geq x^2 + y^2 - 1$ iff $2xy - 2x - 2y + 2 \geq 0$ iff $1 + xy \geq x + y$. Since $x + y > 1$ then $x = 0$ will implicate $y > 1$ (contr. $y \leq 1$). Hence $x \neq 0$. Similarly $y \neq 0$ and $xy > 0$. On the other hand $x, y \leq 1$. And so, $1 < x + y \leq 2$. Hence, $1 + xy \geq x + y$ is always satisfied. □

Obviously, the above inclusions are satisfied for any $\alpha \geq 1$. The corresponding proofs assuming $x \leq y$ (assuming $x + y \leq 1$) are trivial: we have $L = R$, e.g. case (b): since $x + y \leq 1$ then $L = 0$. From $x \geq x^\alpha$ and $y \geq y^\alpha$ it follows that $1 \geq x + y \geq x^\alpha + y^\alpha$. Hence $x^\alpha + y^\alpha - 1 \leq 0$ and $R = 0$.

The proofs related to $x > y$ or $x + y > 1$ (cases (a) and (b), respectively) correspond to the following two inequalities: $(1 - x + y)^\alpha \leq 1 - x^\alpha + y^\alpha$ (for $x > y$) and $(x + y - 1)^\alpha \geq x^\alpha + y^\alpha - 1$ (for $x + y > 1$).

Let consider case (a). Since $y < x$ then $y - x < 0$. Hence $1 + y - x < 1$ and $(1 + y - x)^\alpha \leq 1 + y - x < 1$. Similarly, $y^\alpha < x^\alpha$, $y^\alpha - x^\alpha < 0$ and $1 + y^\alpha - x^\alpha < 1$. Since $x^\alpha > y^\alpha$ then for any increasing $\alpha \geq 1$ the absolute value $| y^\alpha - x^\alpha |$ will be decreasing and hence $1 - | y^\alpha - x^\alpha | = 1 - x^\alpha + y^\alpha$ will be increasing. At the same time the left side is decreasing (see the example table below: $x = 1/2$, $y = 1/3$, here e.g. $R(4)$ is about twice greater than $L(4)$).

Consider now case (b). Since $x + y > 1$ then $x, y \neq 0$ and the obtained value for $x^\alpha + y^\alpha$ may or not be less than 1 (depending on α). Let $x^\alpha + y^\alpha \leq 1$. Then $R = 0$ and $L = (x + y - 1)^\alpha > 0$, since $x + y - 1 > 0$. Otherwise, should be satisfied the following equivalent inequality: $(x + y - 1)^\alpha + 1 \geq x^\alpha + y^\alpha$. Since $0 < x, y \leq 1$ then $0 < x + y \leq 2$. According to the last inequality, assuming $x = y = 1$ we have: $2 = 2$. In any other situation the left side will not be less than the right one. In fact, for any increasing $\alpha \geq 1$ we can obtain: $L(\alpha) \leq R(\alpha)$, case (a) and $L(\alpha) \geq R(\alpha)$, case (b). A more formal treatment is omitted.

α	$L(\alpha)$	$R(\alpha)$
1	1080 / 1296	1080 / 1296
2	900 / 1296	1116 / 1296
3	750 / 1296	1182 / 1296
4	625 / 1296	1231 / 1296

5. Conclusions

Any new t-norm implies some new applications, e.g. such as: introduction of new t-norm based measures or also computations related to the probability of fuzzy events, specification of new commutative and associative copulas, new possibilities to combine criteria in multicriteria decision making (for evaluation the truth degrees of compound formulae), new kind of fuzzy t-equivalence and so on. Fuzzy rough sets have become an important part of modern computer science. It has presented a possibility of generalisation of the notions of lower and upper approximations used in fuzzy rough sets and also of obtaining better such approximations. More formally, the obtained Hamming distance $d(\underline{\mu}, \bar{\mu})$ is decreasing with respect to increasing $\alpha \geq 1$. The so-obtained approximations can be used in very many areas, e.g. such as medical imaging, fuzzy control, data bases, and so on. Any such applications may be topics for further research.

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Iwan Tabakow Dr. Iwan Tabakow received his M.Sc., Ph.D. and D.Sc. degrees as follows: Wroclaw University of Technology (M.Sc., Computer Engineering 1969), The University of Wroclaw (M.Sc., Mathematics 1972), Wroclaw University of Technology (Ph.D. and D.Sc., Computer Science, 1972 and 1975, respectively). His professional experience includes: assistant professor and lecturer (Institute of technical Cybernetics, Wroclaw University of Technology and the Department of Mathematics, The University of Wroclaw: 1973 - 1975), assistant professor, associate professor, and full professor at the Department of Computer Science and Engineering, Technical University of Sofia (1976 - 1984, 1984 - 1996, and 1996 - 1998, respectively), full professor in Computer Science at the Institute of Informatics, Faculty of Computer Science and Management (Wroclaw University of Technology: 1998 - 2014). Dr. Tabakow is currently a lecturer in the following three topics: "Discrete mathematical structures", "Fault Diagnosis of Digital Systems", and "Petri Nets". Areas of main research interests: discrete structures, formal logic, system diagnosis and Petri nets. He has about eighty publications all in the above given areas (including four monographs, journal and refereed conference papers). He was founder, chairman, and/or reviewer of a number of national and international scientific conferences, external examiner, and so on. He is member of Geselschaft für Informatic e.V.Bonn.