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A Mathematical Model for the Rainbow Vertex-Connection Number

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Abstract

The concept of rainbow connection was introduced by Chartrand et al. [2] in 2008. A vertex-colored graph is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection of a connected graph G, denoted by $rvc(G)$, is the minimum number of colors needed to make G rainbow vertex-connected. In this paper, we introduce the first mathematical model of the problem.

Keywords: *Rainbow Vertex Connection, Graph Coloring, Integer Programming, Mathematical Modeling.*

1. Introduction

An edge coloring of a graph is a function from its edge set to the set of natural numbers. A path in an edge colored graph with no two edges sharing the same color is called a rainbow path. An edge colored graph is said to be rainbow connected if every pair of vertices is connected by at least one rainbow path. Such a coloring is called a rainbow coloring of the graph. The minimum number of colors required to rainbow color a connected graph G is called its rainbow connection number, denoted by $rc(G)$. For a basic introduction to the topic, see Chapter 11 in [3].

The concept of rainbow connectivity was recently introduced by Chartrand et al. in [2] as a measure of strengthening connectivity. The rainbow connection problem, apart from being an interesting combinatorial property, also finds an application in routing messages [4]. In addition to being a natural combinatorial measure, rainbow connectivity can be motivated by its interesting interpretation in the area of networking. Suppose that G represents a network (e.g., a cellular network). We wish to route messages between any two vertices in a pipeline, and require that each link on the route between the vertices (namely, each edge on the path) is assigned a distinct channel (e.g. a distinct frequency). Clearly, we want to minimize the number of distinct channels that we use in our network. This number is precisely $rc(G)$ [11].

As one can see, the above rainbow connection number involves edge colorings of graphs. A natural idea is to generalize it to a concept that involves vertex-colorings. In [8], Krivelevich and Yuster are the first to introduce a new parameter corresponding to the rainbow connection number which is defined on a vertex-colored graph. A vertex-colored graph G is rainbow vertex-connected if its every two distinct vertices are connected by a path whose internal vertices have distinct colors. A vertex-coloring under which G is rainbow vertex-connected is called a rainbow vertex-coloring. The rainbow vertex-connection number of a connected graph G, denoted by $rcv(G)$ is the smallest number of colors that are needed in order to make G rainbow vertex-connected. The minimum rainbow vertex-coloring is defined similarly [13].

Rainbow connectivity from a computational point of view was first studied by Caro et al. [4] who conjectured that computing the rainbow connection number of a given graph is NP-Hard. This conjecture was confirmed by Chakraborty et al. [11], who proved that even deciding whether rainbow connection number of a graph equals 2 is NP-Complete. They also conjectured that for every integer $k \geq 2$ to decide whether $rc(G) \leq k$ is NP-Hard. Recently, Ananth and Nasre confirmed the conjecture in [10]. Li and Li [9] showed that for any fixed integer $k \geq 2$ to decide

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whether $rc(G) \leq k$ is actually NP-Complete. For the rainbow vertex-connection number, we have a similar complexity result in [5].

The authors in [4], [11], [8] view rainbow connection number as a "quantifiable" way of strengthening the connectivity property of a graph. Hence, tighter upper bounds on the rainbow connection number for a graph with higher connectivity have been a subject of investigation. There are a number of algorithms that are proposed for the problem. Some of these are Basavaraju's approximation algorithm [1], Deng's polynomial algorithm [6], and Ugurlu and Nuriyeva's heuristic algorithm [12].

To the best of our knowledge, no mathematical model has been proposed for the rainbow vertex-connection problem. In this paper, we present a mathematical model of the rainbow vertex-connection problem.

2. Mathematical Modeling of the Rainbow Vertex-Connection Problem

The graphs we consider are finite, simple, undirected, and loopless. Let $G = (V,E)$ be a graph, where $V=\{v_1, v_2, \ldots, v_n\}$ is the set of vertices, and $E=\{(v_i,v_j) | v_i,v_j \in V\}$ is the set of edges in G such that $|V| = n$ and $|E| = m$.

Let $A = a_{ij}$ be the adjacency matrix of graph G as follows:

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge between i and j,} \\ 0, & \text{otherwise} \end{cases}$$

We define the variables y_i and x_{ij}^{pq} as follows:

$$y_i \in \mathbb{N}$$

and

$$x_{ij}^{pq} = \begin{cases} 1, & \text{if the edge } (v_i, v_j) \text{ is on the } v_p \to v_q, \\ 0, & \text{otherwise,} \end{cases}$$

for all $(v_i,v_j) \in E$, $p = 1,\ldots, n-1$, $q = p+1,\ldots, n.$ The variable y_i represents the color of the vertex $v_i \in V$.

Let v_p and v_q be the source vertex and the target vertex, respectively. The paths between all p,q pairs can be modeled as follows: Since there can be only one outgoing edge in a path from the source,

$$\sum_{i=1}^{n} x_{pi}^{pq} = 1,$$
$$\text{for all } (v_p, v_i) \in E,\ p = 1,\ldots, n-1,\ q = p+1,\ldots, n,$$

and there can be only one incoming edge in a path to the target,

$$\sum_{i=1}^{n} x_{iq}^{pq} = 1,$$
$$\text{for all } (v_i, v_q) \in E,\ p = 1,\ldots, n-1,\ q = p+1,\ldots, n.$$

Since the number of the incoming and outgoing edges are equal for each vertex in a path apart from the source and the target,

$$\sum_{i=1}^{n} x_{ik}^{pq} = \sum_{i=1}^{n} x_{ki}^{pq},$$
$$\text{for all } (v_i, v_k) \in E,\ k = 1,\ldots,n,\ p = 1,\ldots, n-1,\ q = p+1,\ldots, n : k \neq p,q.$$

Theorem 1 Since the internal vertices in a rainbow path connecting vertices v_p and v_q must be colored with different color (v_p and v_q may be colored or not colored with the same color), the following inequality is necessary for the vertex-rainbow problem:

$$\sum_{\substack{i=1 \\ (v_i,v_j)\in E}}^{n-1} \sum_{j=1}^{n} (y_i + y_j) x_{ij}^{pq} \geq 2\left(1 + 2 + \ldots + \left(\left(\sum_{\substack{i=1 \\ (v_i,v_j)\in E}}^{n-1} \sum_{j=1}^{n} x_{ij}^{pq}\right) - 1\right)\right)$$

for all $p = 1,\ldots,n-1$ *and* $q = p+1,\ldots,n.$

Proof: It is clear that the number of edges in a rainbow path connecting vertices v_p and v_q is $\sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ (v_i,v_j)\in E}}^{n} x_{ij}^{pq}$.

Therefore, in this rainbow path, the number of vertices other than v_p and v_q is $\left(\sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ (v_i,v_j)\in E}}^{n} x_{ij}^{pq}\right) - 1$. Since source and destination vertices in each rainbow path may not be colored and the other vertices in a rainbow path must be colored with different color, it is needed $\left(\sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ (v_i,v_j)\in E}}^{n} x_{ij}^{pq}\right) - 1$ different colors to color these internal vertices. Let y_i and y_j be the colors of the endpoints of an edge (v_i,v_j). Then, the following constraint must be satisfied for a rainbow path connecting vertices v_p and v_q:

$$\sum_{\substack{i=1 \\ (v_i,v_j)\in E}}^{n-1}\sum_{j=1}^{n} (y_i + y_j)x_{ij}{}^{pq} \geq 2\left(1+2+\ldots+\left(\left(\sum_{\substack{i=1 \\ (v_i,v_j)\in E}}^{n-1}\sum_{j=1}^{n} x_{ij}{}^{pq}\right)-1\right)\right).$$

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The constraint above can be re-written as follows:

$$\sum_{\substack{i=1 \\ (v_i,v_j)\in E}}^{n-1}\sum_{j=1}^{n} (y_i + y_j)x_{ij}{}^{pq} \geq \left(\left(\sum_{\substack{i=1 \\ (v_i,v_j)\in E}}^{n-1}\sum_{j=1}^{n} x_{ij}{}^{pq}\right)-1\right)\left(\sum_{\substack{i=1 \\ (v_i,v_j)\in E}}^{n-1}\sum_{j=1}^{n} x_{ij}{}^{pq}\right).$$

Consequently, the rainbow vertex-connection problem can be formulated as the following integer nonlinear program:

$$rvc(G) = \min \sum_{i=1}^{n} y_i \tag{1}$$

subject to

$$\sum_{i=1}^{n} x_{pi}{}^{pq} = 1, \tag{2}$$
for all $(v_p, v_i) \in E,\ p = 1,\ldots, n\text{-}1,\ q = p+1,\ldots, n.$

$$\sum_{i=1}^{n} x_{iq}{}^{pq} = 1, \tag{3}$$
for all $(v_i, v_q) \in E,\ p = 1,\ldots, n\text{-}1,\ q = p+1,\ldots, n.$

$$\sum_{\substack{i=1 \\ (v_i,v_k)\in E}}^{n} x_{ik}{}^{pq} = \sum_{\substack{i=1 \\ (v_k,v_i)\in E}}^{n} x_{ki}{}^{pq}$$
for all $k = 1,\ldots, n,\ p = 1,\ldots, n\text{-}1,\ q = p+1,\ldots, n : k \neq p, q.$ (4)

$$\sum_{\substack{i=1 \\ (v_i,v_j)\in E}}^{n-1}\sum_{j=1}^{n} (y_i + y_j)x_{ij}{}^{pq} \geq \left(\left(\sum_{\substack{i=1 \\ (v_i,v_j)\in E}}^{n-1}\sum_{j=1}^{n} x_{ij}{}^{pq}\right)-1\right)\left(\sum_{\substack{i=1 \\ (v_i,v_j)\in E}}^{n-1}\sum_{j=1}^{n} x_{ij}{}^{pq}\right).$$
for all $p = 1,\ldots, n\text{-}1$ *and* $q = p+1,\ldots, n.$ (5)

$x_{ij}{}^{pq} = 0 \vee 1,$ for all $(v_i, v_j) \in E,\ p = 1,\ldots, n\text{-}1,\ q = p+1,\ldots, n.$ (6)

$y_i = k$ for all $(v_i, v_j) \in E,$ where $k \in \mathbb{N}.$ (7)

The proposed model satisfies the rainbow coloring of the graphs and $\max\{y_i \mid v_i \in V\}$ gives the minimum rainbow vertex-connection number. The number of variables and constraints in the model are $O(mn^2)$ and $O(n^3)$, respectively.

3. Conclusions

The rainbow vertex-connection number is a new concept for measuring the connectivity of a graph. In this paper, we present a mathematical model of the rainbow vertex-connection problem. In future works, we aim to investigate the computational results on graphs with various densities.

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